Notes on the stability of dynamic systems and the use of Eigen Values.

Source: Macro II course notes, Dr. David Bessler's Time Series course notes, Azariadis (1999) "Intertemporal Macroeconomics" chapter 4 & Technical Appendix, and Hamilton (1994) "Time Series Analysis" chapter 1 & Mathematical Review Appendix.

Prepared by Ariel Vitale 2003

Let the first-order linear non-linear non-homogeneous system of difference equations be in general form:[1]

$$z_{t+1} = Az_t + f_{t+1} \qquad (1)$$

where A is a $n \times n$ matrix of time invariant coefficients and *z* and *f* are $n \times 1$ vector of dated variables. The variable z_t is typically the state vector of the system at time *t*; f_t is a vector of (possibly time-dependent) forcing terms often thought of as exogenous. For illustrative purposes we will work with a 2 x 2 system of the form:

$$\begin{aligned} x_{t+1} &= ax_t + by_t + \bar{x} \\ y_{t+1} &= cx_t + dy_t + \bar{y} \end{aligned} \qquad (2)$$

or,

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}. \qquad (3)$$

We can write the general solution to any linear system like in (1) in the form of a complementary homogeneous system plus the particular solution, or:

$$z_t^g = z_t^C + z_t^P \qquad (4)$$

This result (4), allows us to solve for the equilibrium of the system in (1) in two stages: we start with the homogeneous system $z_{t+1} = Az_t$, and complete the task by finding the particular solution to the full system. Particular solutions are easy to find when the

---

[1] Alternatively the analysis can be done in terms of differential equations, changing notation $x_t \to x(t)$. Recall by Kolmogorov's continuity theorem, that for the discrete process $\{x_t\}_{t \geq 0}$ and $\forall T > 0$ if there exist positive constants α, β, D s.t. $E\left[\left|x_t - x_s\right|^\alpha\right] \leq D \cdot \left|t - s\right|^{1+\beta};\ 0 \leq s, t \leq T$, then there exists a continuous version of *x, x(t)*.

forcing terms are time invariant because in the steady state $x_{t+1} = x_t = x$ and $y_{t+1} = y_t = y$. Thus,

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \; where \; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5}$$

regrouping terms,

$$(I - A)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \; where \; I = Identity \; Matrix \tag{6}$$

and,

$$\begin{pmatrix} x \\ y \end{pmatrix} = (I - A)^{-1}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \tag{7}$$

that is iff $(I - A)$ is invertible, i.e. non-singular. Plugging (6) into (3):

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = A\begin{pmatrix} x_t \\ y_t \end{pmatrix} + (I - A)\begin{pmatrix} x \\ y \end{pmatrix} \tag{8}$$

Regrouping terms, we get the homogeneous linear system of first-order difference equations:

$$\begin{pmatrix} x_{t+1} - x \\ y_{t+1} - y \end{pmatrix} = A\begin{pmatrix} x_t - x \\ y_t - y \end{pmatrix} \tag{9}$$

Assume we know the initial condition $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, then

$$\begin{pmatrix} x_{t+1} - x \\ y_{t+1} - y \end{pmatrix} = A^t\begin{pmatrix} x_0 - x \\ y_0 - y \end{pmatrix}. \tag{10}$$

Hence, the system converges to the steady state iff $\lim_{t \to \infty} A^t = 0$. This can be determined through a spectral or eigenvalue representation of A^t. To do this we need to use Jordan decomposition, which requires that matrix A^t has $s \leq n$ linearly independent eigenvectors, i.e. <u>distinct</u> eigenvalues (could have some but <u>never</u> all eigenvalues repeated). The Jordan decomposition of A^t is:

$$A^t = e\Lambda^t e^{-1} \tag{11}$$

where e is the matrix of eigenvectors and Λ is the matrix of eigenvalues, a diagonal matrix with the eigenvalues along the principal diagonal and zeros elsewhere. For the 2×2 system:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{12}$$

and because Λ is diagonal, it must be:

$$\Lambda^t = \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix}, \text{ where } \lambda_1^t \neq \lambda_2^t \tag{13}$$

The eigenvalues are obtained as follows. Without loss of generality drop the time superscript in (11), then it must be that:

$$A \times e = \Lambda \times e \tag{14}$$

define λ as the vector of eigenvalues and I an $n \times n$ identity matrix, then $\Lambda = \lambda \times I$, and

$$A \times e = \lambda \times I \times e \Rightarrow (A - \lambda \times I) \times e = 0 \tag{15}$$

Thus, iff e is non-singular, then it must be that:

$$\det(A - \lambda \times I) = 0 \tag{16}$$

(16) is known as the characteristic equation from where we obtain the eigenvalues. Notice, that for the 2 x 2 system:

$$\det(A - \lambda \times I) = \det\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = 0$$

$$\Leftrightarrow \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0 \tag{17}$$

$$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$
$$\Rightarrow ad - a\lambda + \lambda^2 - d\lambda - bc = 0$$
$$\Rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

where $(a+d) = Trace(A) = T$ and $(ad-bc) = \det(A) = D$. Then, the characteristic equation can be re-written in terms of the T and D as follows:

$$\lambda^2 - T\lambda + D = 0 \tag{18}$$

Solving the quadratic equation for the eigenvalues:

$$\lambda_1 = \frac{T+\sqrt{T^2-4D}}{2} \quad ,and \ \lambda_2 = \frac{T-\sqrt{T^2-4D}}{2} \tag{19}$$

λ_1 is the largest eigenvalue and λ_2 is the smallest eigenvalue. Also, $\lambda_1 + \lambda_2 = T$ and $\lambda_1 \times \lambda_2 = D$. Recall that, $\Lambda^t = \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix}$, then we can assess the local stability properties of the planar system from the discriminant $\Delta = T^2 - 4D = 0$ (a parabola) of the characteristic equation:

1) Iff $T^2 > 4D$ then both eigenvalues are real and distinct.

2) Iff $T^2 = 4D$ then both eigenvalues are equal and matrix *A* cannot be diagonalizable and we need another method to find the eigenvalues.

3) Iff $T^2 < 4D$, then both eigenvalues are distinct and complex conjugates:

$$\lambda_1 = \frac{T+i\sqrt{4D-T^2}}{2} \quad ,and \ \lambda_2 = \frac{T-i\sqrt{4D-T^2}}{2} \tag{20}$$

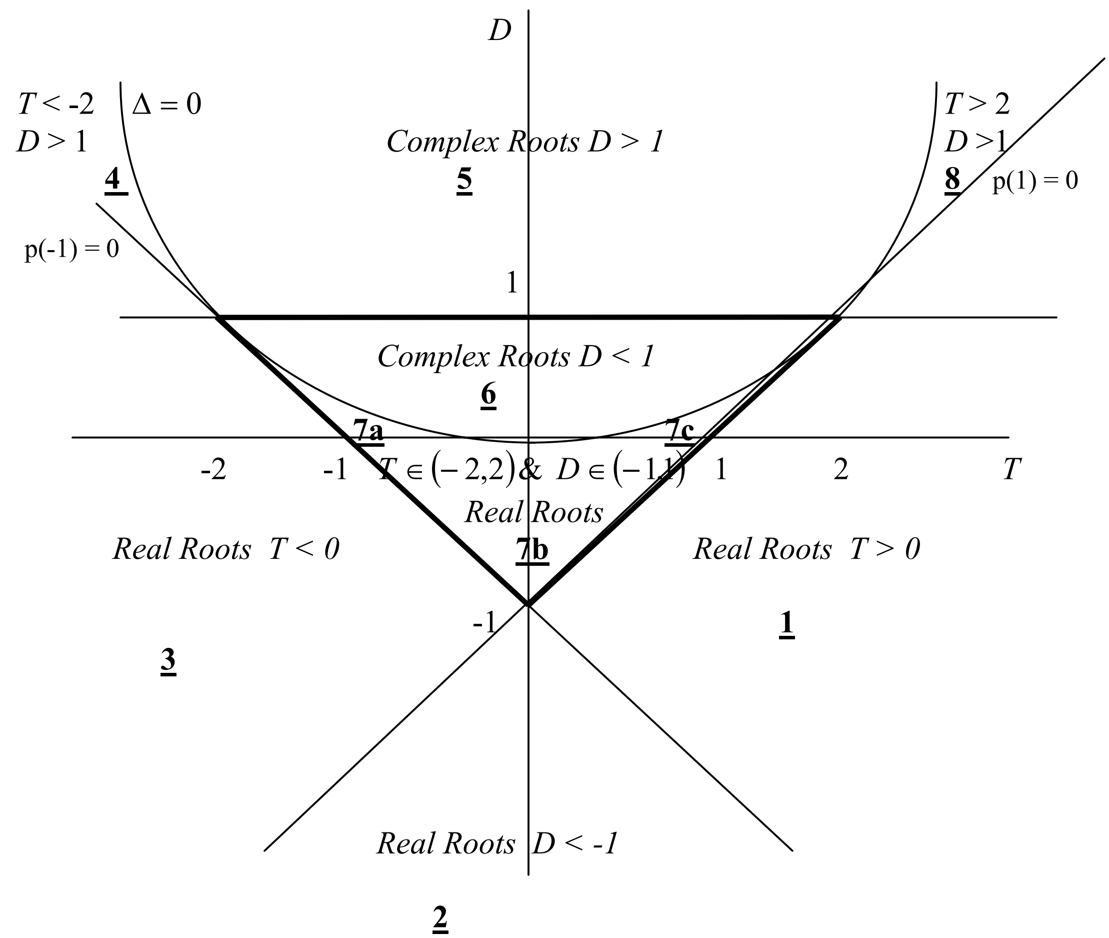
Also:

A) Iff $|\lambda_i| < 1 \ \forall i$, the modulus of the eigenvalues lie within the unit circle (SINK), i.e. the steady state is stable as $\lambda_i^t \to 0$ as $t \to \infty$.

B) Iff $|\lambda_i| > 1 \ \forall i$, the modulus of the eigenvalues lie outside the unit circle (SOURCE), i.e. the steady state is unstable as $\lambda_i^t \to \pm\infty$ as $t \to \infty$.

C) Iff $\exists i \ s.t. \ |\lambda_i| > 1$, there exists some eigenvalue which modulus lies outside the unit circle (but not all) (SADDLE), then the steady state is neither stable nor unstable. Only a very specific set of initial conditions will take the system to the steady state along the saddle path (stable manifold) driven by the stable eigenvalues.

Finally factorizing the characteristic equation *p*, we obtain the straight lines $p(1) = 0 \wedge p(-1) = 0$, which jointly with the discriminant parabola divides the plane in each regions for the characterization of the stability properties of the system in terms of the eigenvalues (roots), or equivalently from the trace (*T*) and the determinant (*D*) of the coefficient matrix *A* (see graph below):



Real eigenvalues zones:
Region 1: Saddle,
Region 2: Source, oscillatory divergence,
Region 3: Saddle,
Region 4: Source, monotone divergence,
Region 7: Sink,
7a & 7c – monotone convergence,
7b – oscillatory convergence,
Region 8: Source, monotone divergence,

Complex eigenvalues (conjugate pairs) zones:
Region 5: Source, oscillatory divergence,
Region 2: Sink, oscillatory convergence.

FINAL REMARK: For the nonlinear case, we linearized the system using Taylor's expansion and finding the Jacobian *(J)* evaluated at the steady state. Then we can proceed with the previous analysis changing A for J.