AGEC 642
Problem Set #2

Problem sets are not graded, but must be completed before taking some MTs.
When you have completed the PS, you may request the answer key.

A word of advice: This problem set is longer than #1. Focus on the big issues: the problem set up, economic interpretation, etc. You should not need to spend hours and hours on algebra. Where you see the words "economically interpret," you should especially build on the intuition provided by Dorfman as covered in Lecture 5. Questions 1 and 2 can be answered using material from Lecture 3. For other parts of the problems you'll use material covered in lecture 5 and a little bit of 6.

1. (Lecture 3) ***Formalize an optimal control problem based on one of the three problems (a, b, or c) presented in question 1 of PS#1***. *You do not need to provide an extremely detailed formulation, but I do want some degree of specificity. For example, you don't need to write down a specific functional form for a production function such as* $y=ax_1+bx_2^2+\ldots$*, but you could specify a general production relationship,* $y=f(x_1,x_2,z_1,z_2)$*, and a profit function such as* $\pi=py-c(x_1,x_2,z_1)$*. To get a sense of what I'm looking for, many papers have a section called "The Model" which should provide some guidance. You should be able to be creative and original.*

   a. Specify the problem as a simple optimal control problem. Identify (1) the time horizon, (2) the benefit function, (3) the objective function, (4) the state variables, (5) the control variables, (6) the state equation(s), (7) the initial and terminal conditions, and (8) any other constraints that you feel are important.

   b. Formally state the optimization problem. i.e. $\max_{z_t} \int u(\cdot)\,dt \;\; s.t. \;\; \dot{x} = f(\cdot)$ and, ...

      State the Hamiltonian, the first-order conditions and the transversality condition(s) of your problem.

   c. Using the problem specification and the first-order conditions, economically interpret the first order condition with respect to the control variable.

2. (Lecture 4) You have x_0 dollars in the bank, which earns interest at the rate of δ percent per year, compounded continuously. Your instantaneous utility is $ln(z_t)$, where z_t is the instantaneous rate at which you spend money from your account. You want to maximize your total utility over the period 0 to T without discounting. You must have \$$x_T$ in your bank account at time T to pay off a loan.

   a. Provide a formal statement of the optimization problem, specifying your objective function and all constraints.

   b. Write out the Hamiltonian, the first order and transversality conditions, clearly identifying each.

   c. Solve for the optimal paths of z, x, and λ. Your mathematical derivation should carefully explain steps, making sure that your derivation is easy to follow. At the end, you should have closed-form expressions for z, x, and λ with only parameters on the right-hand side.
   Ans: $z_t = \dfrac{e^{\delta t}\left(x_0 - e^{-\delta T} x_T\right)}{T}$, $\lambda_t = \left(\dfrac{T}{x_0 - e^{-\delta T} x_T}\right) e^{-\delta t}$, $x_t = \left(\dfrac{(T-t)\,x_0 + t e^{-\delta T} x_T}{T}\right) e^{\delta t}$

3. (Lecture 5) You have a contract to deliver $\bar{x}$ units of output, but there is no specified delivery date. Your goal is to minimize the cost (*no discounting here*). There is one input into your production function, labor, which we denote z_t. Each unit of labor creates output at the rate of 1 unit per hour. Labor can be varied continuously over time. The more labor you use at any point in time, the more costly it becomes. Specifically, your labor cost per hour is $az_t + bz_t^2$. You also consider the cost of your management time, which is worth \$$c$ per hour. For example, if you held your labor constant at 3 units for 1 hour, the cost to you would be $a \cdot 3 + b \cdot 3^2 + c$.

   a. Provide a formal statement of the optimization problem, defining the state and control variables, the objective function, and specifying all constraints of the problem.

   b. Write out the Hamiltonian, first-order conditions and the transversality condition.

   c. Solve for the optimal trajectory for labor, inventory and your co-state variable, circling the final results. (Hint: $z_t = \sqrt{c/b}$ for all t. Make sure you solve for all unknowns, including T to find $T = \bar{x}\left(\dfrac{b}{c}\right)^{0.5}$.)

   d. Economically interpret the solution, answering each of the questions below.
      i. Give an economic interpretation of each of the FOCs. That is, for each FOC, explain WHY, economically, the equality makes sense.
      ii. In the optimal rule, z is constant. Does this make economic sense? WHY or why not?
      iii. How does $\bar{x}$ affect the optimal policy at each point in time? WHY does this make sense?

4. (Lecture 6) For this problem you will work with the optimal trajectories of the fisheries management problem in section V of Lecture 3 using a *current value* Hamiltonian.

   a. Starting with the current-value Hamiltonian, provide a clear and succinct derivation of the equations and inequalities needed to draw a phase diagram describing the optimal management plan in x–μ space, where μ is the current-value co-state variable.

   b. Draw the phase diagram, labeling all axes and curves.

   c. Consider the two sides of the saddle path that lead to the equilibrium; in 2-4 sentences discuss why it is optimal for x and μ to change as they do along the saddle paths.

   d. Derive Hartwick's approximation of Hicksian income (Lecture #6) for this problem and explain the meaning of this result. Your answer should include your derivation (3-5 lines) and a one-sentence explanation explaining why the units make sense.

5. (Lecture 5) (Adapted from Chiang problem 9.3.1) Output, y_t, is a function of your capital stock, x_t, $y_t=Ax_t^{\alpha}$ with $\alpha<1$. Utility is an increasing function of withdrawals from the stock, z_t, $U=\tilde{u}-\frac{z^{-b}}{b}$, where $\tilde{u}$ is a constant and $b>0$. The capital stock depreciates at the rate δx_t but can be increased by the portion of the output that is not consumed, i.e., $\dot{x}_t = y_t - z_t - \delta x_t$. Your objective is to maximize the discounted present value of future utility using discount rate r over an infinite horizon. (Note that you do *not* have to completely solve this problem)

   a. Formally state the optimization problem.

   b. Using the current-value Hamiltonian, apply the maximum principle (i.e., what are the necessary conditions for a maximum, including the transversality condition. For the transversality condition you'll need to note that $\lambda=\mu e^{-rt}$).

   c. Take the text from the quotations from Dorfman in Lecture 5 and explain how those key quotes apply to this problem. Then, in your own words, explain the economic meaning of each of the maximum conditions.

   d. Draw a phase diagram in the x-z space.

   Hint: in the correct answer there is a 1:1 mapping between z and μ.

Reminder: I encourage students to help each other, but keep in mind that the purpose of the problem sets is to help you learn the material and prepare for the associated mastery tests. You will learn the material best if you have worked through the problem set independently; simply seeing someone else's answer is unlikely to be helpful.
