5. The intuition behind optimal control as explained by Dorfman (1969) & the current value Hamiltonian

AGEC 642 - 2024

The purpose of this lecture is to help us understand the intuition behind the optimal control framework. We draw first on Dorfman's seminal article in which he explained OC to economists. I strongly encourage you to refer to the original article (http://www.jstor.org/stable/1810679) as you go through these notes.

(For this lecture, I will use Dorfman's notation, so k is the state variable and x is the choice variable)

A. *The problem*

Dorfman's problem is to maximize

(1) $$W\left(k_t,\vec{x}\right)=\int_t^T u\left(k,x,\tau\right)d\tau$$

where $\vec{x}$ is the stream of all choices made between t and T, and $u(\cdot)$ is the utility function that indicates the rate per period that the planner gains utility from the choices, x, the state, k, at time τ.

The state equation is

$$\dot{k}=\frac{\partial k}{\partial t}=f\left(k,x,t\right)$$

B. *Step 1. Divide time into two pieces*

Dorfman's first step is to divide the time from t to T into two pieces: from t to $t+\Delta$ and from $t+\Delta$ to T. If Δ is small, then there is little loss of accuracy if we linearize utility over the interval from t to $t+\Delta$, i.e., assume that k, x, and $u(k,x,t)$ are constant over this interval. Technically, all the "=" signs below should be replaced by "≈" signs, but we will assume the approximation error is trivial. Hence, we rewrite

$$W\left(k_t,\vec{x}\right)=\Delta u\left(k,x_t,t\right)+\int_{t+\Delta}^T u\left(k,x,\tau\right)d\tau .$$

First, let's look just at this second term. If we assume that we maximize over the second interval from $t+\Delta$ to T, then we can eliminate the control variable, $\vec{x}$, from the second term to obtain

(2) $$V^*\left(k_{t+\Delta},t+\Delta\right)=\max_{\vec{x}} W\left(k_{t+\Delta},\vec{x},t+\Delta\right)=\int_{t+\Delta}^T u\left(k^*,x^*,\tau\right)d\tau ,$$

where k^* and x^* are the optimal paths of the state and control variables, respectively.

Using this value function, $V^*\left(k_{t+\Delta},t+\Delta\right)$ in (2), we can write the value of the stream of welfare starting at time t with assets k_t and choosing a constant value x_t for the period from t to $t+\Delta$ as follows:

(3) $$V\left(k_t,x_t,t\right)=\Delta u\left(k_t,x_t,t\right)+V^*\left(k_{t+\Delta},t+\Delta\right).$$

The reason we multiply $u(\cdot)$ by Δ is that we are assuming that $u(\cdot)$, the rate per period at which utility is being generated, is constant for the period from t to $t+\Delta$. Hence,

$\Delta u(k_t, x_t, t)$ is the amount of utility accumulated over this short period. If Δ=1, then $\Delta u(\cdot)$ is one period's worth of utility. If Δ=½, then $\Delta u(\cdot)$ is half a period's worth of utility.

Similarly, we know that $k_{t+\Delta} = k_t + \int_t^{t+\Delta} f(k, x_t, t)\,dt$, i.e., k_t plus the changes in k_t that occur from *t* to *t*+Δ. But if $f(k, x_t, t)$ is assumed to be constant from *t* to *t*+Δ, then $k_{t+\Delta} = k_t + \Delta f(k_t, x_t, t)$.

Note that the *V*(·) on the LHS of (3) is different from the $V^*(\cdot)$ on the RHS in (2). *V*(·) does not have a * since it is not necessarily at the optimum; it includes x_t as an argument. When we write $V^*(k_{t+\Delta}, t+\Delta)$ it means that it is evaluated at the optimum value of *x* so that it is only *k* and *t*. In contrast, $V(k_t, x_t, t)$ can be evaluated at all possible values of x_t, including suboptimal values, allowing us to maximize *V*(·) over x_t.

C. Step 2. Evaluate the FOC w.r.t. the control variable, x_t

The optimum choice, x_t, of (3) can be found using standard tools of calculus. Dorfman takes the FOC, directly with respect to the choice variable x_t

(4) $$\frac{\partial V(k_t, x_t, t)}{\partial x_t} = \Delta \frac{\partial}{\partial x_t} u(k_t, x_t, t) + \frac{\partial}{\partial x_t} V^*(k_{t+\Delta}, t+\Delta) = 0.$$

We can then rewrite the second term

(5) $$\frac{\partial V^*}{\partial x_t} = \frac{\partial V^*}{\partial k_{t+\Delta}} \frac{\partial k_{t+\Delta}}{\partial x_t}.$$

Since we assume that the interval Δ is quite short, we can approximate the state equation

$k_{t+\Delta} = k_t + \dot{k}\Delta = k_t + f(k, x_t, t)\Delta$

so that

(6) $$\frac{\partial k_{t+\Delta}}{\partial x_t} = 0 + \frac{\partial f}{\partial x_t}\Delta.$$

Dorfman then substitutes (6) into (5), and also writes $\frac{\partial V^*}{\partial k_t} = \lambda_t$, so that (4) can be rewritten

(7) $$\Delta \frac{\partial u}{\partial x_t} + \lambda_{t+\Delta} \Delta \frac{\partial f}{\partial x_t} = 0.$$

> Note that (7) and the relationship between *V'* and *λ* can also be derived if we start with a Lagrangian,
>
> $L = u(k_t x_t, t)\Delta + V^*(k_{t+\Delta}, t+\Delta) - \lambda_{t+\Delta}\left(k_{t+\Delta} - \left(k_t + f(k, x_t, t)\Delta\right)\right).$
>
> The FOCs would be,

$$\Delta\frac{\partial u}{\partial x_t}+\lambda_{t+\Delta}\Delta\frac{\partial f}{\partial x_t}=0\text{, and}$$

$$\frac{\partial V^*(\cdot)}{\partial k_{t+\Delta}}=\lambda_{t+\Delta}.$$

This confirms what we already know: that λ is the value of marginally relaxing the constraint, i.e., the change in V^*, that would be achieved by a marginal increase in $k_{t+\Delta}$. Hence, V' and λ are equivalent!

Cancelling Δ in (7) and then taking the limit as $\Delta\to 0$ so that $\lambda_{t+\Delta}=\lambda_t$, we obtain

(8) $$\frac{\partial u}{\partial x_t}=-\lambda_t\frac{\partial f}{\partial x_t}.$$

This is the first of the optimality conditions of the maximum principle, (i.e., $\partial H/\partial z=0$).

Dorfman (822-23) provides a clear and succinct economic interpretation of this term:

> [Equation (8)] says that the choice variable at every instant should be selected so that the marginal immediate gains are in balance with the value of the marginal contribution to the accumulation of capital.

Put another way, the choice variable should be increased as long as the marginal immediate benefit is greater than the marginal future costs of that increase.[1] In problems where the choice variable is discrete or constrained, it may not be possible to actually achieve the equi-marginal condition, but the intuition remains the same.

So now we've got a nice intuitive explanation for the first of the maximum conditions:

> ***The central principle of dynamic optimization is that optimal choices are made when a balance is struck between the immediate and future marginal consequences of our choices.***

D. A simple problem to clarify the meaning of λ and $\dot{\lambda}$

(This section is presented in an online video that can be seen via the AGEC 642 website).

Before returning to Dorfman's analysis, let's consider a simple dynamic optimization problem that provides some basic intuition.

Suppose that our optimization problem is as follows

$$\max_{x_t}\int_s^T k_t\left(ax_t-\frac{b}{2}x_t^2\right)dt \qquad s.t\ \dot{k}_t=c\cdot k_t$$

where s is an arbitrary starting time less than T.

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[1] Obviously, in some problems, the intuition might instead be that the choice variable should be increased as long as the marginal <u>future</u> benefit is greater than the marginal <u>present</u> costs of that increase.

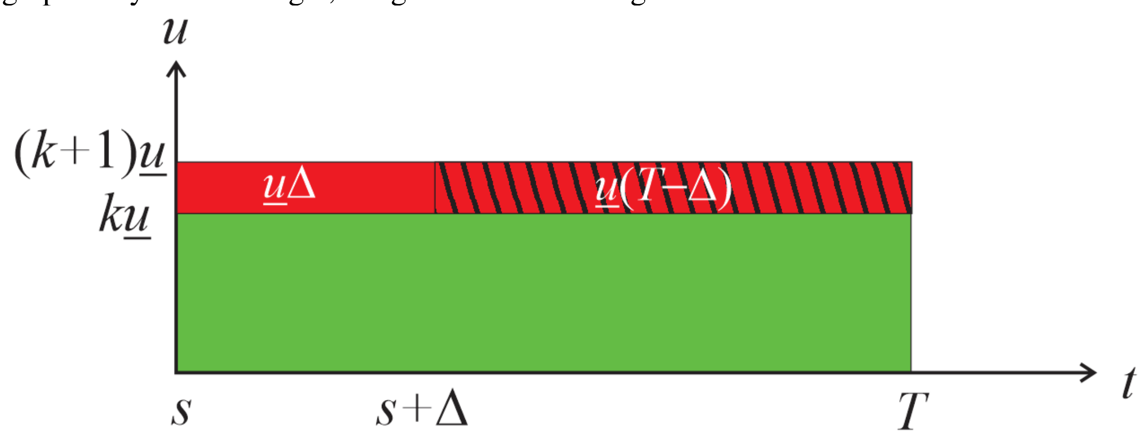
The Hamiltonian and first order conditions would be

$$H = k_t\left(ax_t - \frac{b}{2}x_t^{\ 2}\right) + \lambda_t c \cdot k_t$$

1^{st} FOC: $H_x = k_t\left(a - bx_t\right) = 0$ 2^{nd} FOC: $H_k = -\dot{\lambda}_t = \left(ax_t - \frac{b}{2}x_t^{\ 2}\right) + \lambda_t c$.

From the first FOC, we see that $x_t^* = \frac{a}{b}$ and the benefits at every point in time are simply $u = k_t\left(\frac{a^2}{b} - \frac{a^2}{2b}\right)$, which we will write as $k_t\underline{u}$. On the surface, this is not a very interesting problem, but it actually serves as a helpful case for understanding some of the economic principles in optimal control.

Consider first the case when c=0. In that case, k is constant, which means that the value of the objective function is simply $V_s = \int_s^T k\underline{u}dt = \left(T - s\right)k\underline{u}$, which can be presented graphically as a rectangle, the green area in the figure below.



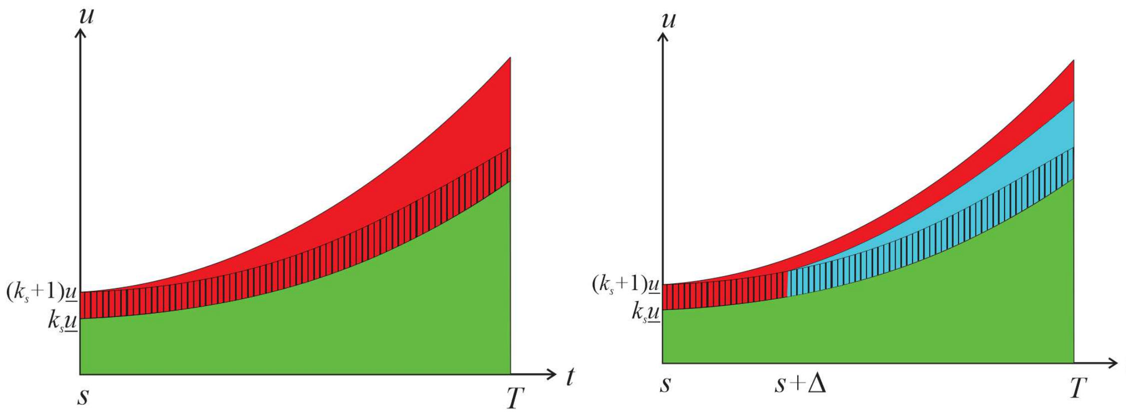
Now let's use this example to understand the marginal value of a unit of the state variable. If we added one more unit of k at the starting time, t=s, then an additional unit of $\underline{u}$ would accrue over the entire time horizon, the red area from s to T. The marginal value of k at time s, λ_s, therefore, captures the value that can be obtained from a marginal increment to k across the entire planning horizon. We also can see this by taking the derivative of V_s with respect to k_s: $\lambda_s = \frac{\partial V_s}{\partial k_s} = (T - s)\underline{u}$.

Now imagine that the marginal unit of k arrived slightly later, at t=s+Δ. In this case, the benefits would accrue for less time, and the red cross-hatched area in the figure equals $\lambda_{t+\Delta}$. Clearly in this case $\lambda_{s+\Delta} < \lambda_s$ and the difference between the two is $\underline{u}\cdot\Delta$ or

$$\lambda_{s+\Delta} - \lambda_s = -\Delta\underline{u} \qquad \text{or} \qquad \lim_{\Delta\to 0}\left(\lambda_{s+\Delta} - \lambda_s\right)/\Delta = \dot{\lambda}_s = -\underline{u}\,.$$

Hence, <u>the rate at which λ falls is exactly equal to the rate at which k gives rise to current benefits.</u> That is, $\dot{\lambda}_t = -\underline{u}$.

Now, assume that $c>0$, so that the resource grows geometrically over time at the rate c. It still holds that $x_t^* = a/b$ for all t, but in this case the value of the objective function is an area under a curve that is increasing at the rate c, the green area in the figure below.



A one unit increase in k at time s would add the full red area in the graph on the left since that additional unit grows at the rate c. Hence, λ_s captures two separate effects. First, the marginal value of an increase in the stock at time s includes the increment to direct benefits over the entire time horizon, $\underline{u}\cdot(T-s)$ indicated by the vertically marked section. Second, the extra unit at time s leads to additional growth in k, which creates more value, which is indicated by the red section in the figure that is above the vertically marked area.

As above, in the figure on the right, we see that if the increment to k happens at $s+\Delta$ instead of at s it is not as valuable. i.e., $\lambda_{s+\Delta} < \lambda_s$ In this case, there are two reasons that λ is declining over time. First, if the increment in k arrives at $s+\Delta$ instead of at s, the direct addition to the objective function, $\underline{u}$ at every moment, does not last as long. This is seen in the fact that the vertically marked area over the full time horizon from s to T is smaller if the increment is obtained in $s+\Delta$, equal to the effect when $c=0$ discussed above. Second, as can also be seen in the figure, since k grows over time, if an increment arrives later there is less time to benefit from that growth – the area above the vertically marked area in the figure, is smaller if started at $s+\Delta$ than if that extra unit had arrived at s. It turns out that in this case, the two effects reduce to $-\dot{\lambda} = \underline{u} + \lambda_t c$.

Hence, <u>the rate at which λ falls is exactly equal to the rate at which k gives rise to current benefits plus the rate at which capital itself gives rise to future benefits.</u>

E. Step 3. Look at the value of λ_t by taking $\partial V^*/\partial k_t$

Now, we return to the general case and Dorfman's derivation. Dorfman now assumes that the optimal choice of x has been made over our short interval, t to Δ.

$$V^*(k_t,t) = u(k_t, x_t^*, t)\Delta + V^*(k_{t+\Delta}, t+\Delta)$$

Differentiating this expression w.r.t. k and substituting λ_t for $\partial V^*(k_t,t)/\partial k_t$, we get

$$\lambda_t = \Delta \frac{\partial u}{\partial k_t} + \frac{\partial}{\partial k_t} V^*\left(k_{t+\Delta}, t+\Delta\right)$$

$$\lambda_t = \Delta \frac{\partial u}{\partial k_t} + \frac{\partial V^*\left(k_{t+\Delta}, t+\Delta\right)}{\partial k_{t+\Delta}} \frac{\partial k_{t+\Delta}}{\partial k_t}$$

$$\lambda_t = \Delta \frac{\partial u}{\partial k_t} + \lambda_{t+\Delta} \frac{\partial k_{t+\Delta}}{\partial k_t}$$

Since this is over a short period, we can approximate

$\lambda_{t+\Delta} = \lambda_t + \dot{\lambda}_t \Delta$ and $k_{t+\Delta} = k_t + \dot{k}_t \Delta$, so that $\frac{\partial k_{t+\Delta}}{\partial k_t} = 1 + \Delta \frac{\partial f}{\partial k_t}$

Hence,

$$\lambda_t = \Delta \frac{\partial u}{\partial k} + \left(\lambda_t + \dot{\lambda}\Delta\right)\left(1 + \Delta \frac{\partial f}{\partial k}\right)$$

$$\cancel{\lambda_t} = \Delta \frac{\partial u}{\partial k} \cancel{+ \lambda_t} + \dot{\lambda}\Delta + \lambda_t \Delta \frac{\partial f}{\partial k} + \dot{\lambda}\Delta^2 \frac{\partial f}{\partial k}$$

$$0 = \cancel{\Delta} \frac{\partial u}{\partial k} + \dot{\lambda}\cancel{\Delta} + \lambda_t \cancel{\Delta} \frac{\partial f}{\partial k} + \dot{\lambda}\Delta^{\cancel{2}} \frac{\partial f}{\partial k}$$

or,

$$-\dot{\lambda} = \frac{\partial u}{\partial k} + \lambda_t \frac{\partial f}{\partial k} + \dot{\lambda}\Delta \frac{\partial f}{\partial k}.$$

Taking the limit at $\Delta \rightarrow 0$, the last term falls out and we're left with

(9) $$-\dot{\lambda} = \frac{\partial u}{\partial k} + \lambda \frac{\partial f}{\partial k}$$

which is the second maximum condition, $-\dot{\lambda} = \partial H / \partial k$.

Dorfman (p. 821) offers 3 ways to think about the economic intuition behind this equation.

> To an economist, it $\left[\dot{\lambda}\right]$ is the rate at which the capital is appreciating. $-\dot{\lambda}$ is therefore the rate at which a unit of capital depreciates at time *t*. … In other words, [1] a unit of capital loses value or depreciates as time passes at the rate at which its potential contribution to profits becomes its past contribution. … [or] [2] Each unit of the capital good is gradually decreasing in value at precisely the same rate at which it is giving rise to valuable outputs. [3] We can also interpret $-\dot{\lambda}$ as the loss that would be incurred if the acquisition of a unit of capital were postponed for a short time [which at the optimum must be equal to the instantaneous marginal value of that unit of capital].

So, we see that since the value of the capital stock at the beginning of the problem is equal to the sum of the contributions of the capital stock across time. As we move across time, therefore, the capital stock's ability to contribute to *V* is "used up." This is the general principle that is shown in the example in the previous section.

Better intuition for [2] above can be found by looking at λ_t and $\lambda_{t+\Delta}$. We know that

$$\lambda_t = \frac{\partial V\left(k_t, x_t, t\right)}{\partial k_t} = \frac{\partial u\left(k_t, x_t, t\right)}{\partial k_t}\Delta + \frac{\partial V^*\left(k_{t+\Delta}, t+\Delta\right)}{\partial k_{t+\Delta}}\frac{\partial k_{t+\Delta}}{\partial k_t}$$

which, since $\frac{\partial k_{t+\Delta}}{\partial k_t} = 1 + f_k\Delta$, can be written

$$\lambda_t = \frac{\partial V^*\left(k_t, t\right)}{\partial k_t} = \frac{\partial u\left(k_t, x_t, t\right)}{\partial k_t}\Delta + \frac{\partial V^*\left(k_{t+\Delta}, t+\Delta\right)}{\partial k_{t+\Delta}}\left(1 + f_k\Delta\right).$$

We can then subtract

$$\lambda_{t+\Delta} = \frac{\partial V^*\left(k_{t+\Delta}, t+\Delta\right)}{\partial k_{t+\Delta}}$$

To obtain the difference between $\lambda_{t+\Delta}$ and λ_t:

$$\lambda_{t+\Delta} - \lambda_t = \left(-\frac{\partial u\left(k_t, x_t, t\right)}{\partial k_t} - \frac{\partial V^*\left(k_{t+\Delta}, t+\Delta\right)}{\partial k_{t+\Delta}} f_k\right)\Delta . \qquad (10)$$

So, we see that the change in λ is composed of two parts, the utility that you get during that period from t to t+Δ, and the marginal value of marginal contribution of k to increasing k itself. This relationship shows how λ must evolve over time along the optimal path.

The key point here is that along its optimal trajectory, λ will take into account all the ways that the marginal unit of k leads to value and how that is used up or created over time.

F. Step 4. Summing up

Hence, each of the optimality conditions associated with the Hamiltonian has a clear economic interpretation.

Let $H = u(k,x,t) + \lambda_t f(k,x,t)$

FOC	Equation	Interpretation
Choice	$\frac{\partial H}{\partial x} = 0$ $\frac{\partial u(\cdot)}{\partial x} + \lambda_t \frac{\partial f(\cdot)}{\partial x} = 0$	Finds the optimal balance between current welfare and future consequences.
State	$\frac{\partial H}{\partial k} = -\dot{\lambda}$ $\frac{\partial u(\cdot)}{\partial k} + \lambda_t \frac{\partial f(\cdot)}{\partial k} = -\dot{\lambda}$	The marginal value of the state variable is decreasing at the same rate at which it is generating benefits. or Along the optimal path, the loss that would be suffered if we delayed acquisition of a marginal unit of capital for an instant must equal the instantaneous marginal value of that unit of capital.
Co-state	$\frac{\partial H}{\partial \lambda} = \dot{k}$ $f(\cdot) = \dot{k}$	The state equation must hold.

II. A word about discounting

Discounting: Recall that if r is the annual *rate* of discount, then $(1+r)^{-T}$ is the discount *factor* applied to benefits or costs T years in the future. If we break each year into n periods, then the periodic discount rate becomes r/n so over n periods (i.e., a year) the one-year discount factor becomes $(1+r/n)^{-n}$. As $n \to \infty$, this converges to e^{-r}, the continuous-time discount factor.

Consider a modification of Dorfman's problem with the assumption that we will maximize the <u>present value</u> of $u(k,x,t)=e^{-rt}w(k,x)$ over the interval 0 to T, i.e.,

$$W = \int_0^T e^{-rt} w(k,x)\,dt$$

This is a restrictive specification of (1) in which $u(k,x,\tau) = e^{-r\tau} w(k,x)$, so the optimality conditions must still hold. The Hamiltonian now is

(11) $H = e^{-rt} w(k,x) + \lambda_t f(k,x,t)$.

The interpretation of λ_t is the same: it is a measure of the contribution to W of a marginal increase in k in period t. However, because of discounting we know that there is additional pressure for λ_t to fall over time. If W_t is the present value (back to year zero) of all the benefits from t to T, then because of discounting W_t will tend to be much smaller far in the future than it is for *t* close to zero. Correspondingly, $\partial W_t / \partial k_t = \lambda_t$ will also tend to fall over time.

Hence, the value of λ_t is influenced by two effects: the current (in period *t*) marginal value of *k*, which could either be increasing or decreasing, and the discounting effect, which is always pushing λ_t toward zero. Hence, even if the marginal value of capital is increasing over time (in current dollars), λ might be falling. Because of these two factors, it often happens that the economic meaning of λ_t is not easily seen – is it falling because *k* is becoming less valuable or simply because of discounting?

Hence, when using a discounted optimization problem, it is almost always preferable to use a modified specification, what is called the *Current Value Hamiltonian*.

A. The Current Value Hamiltonian

We begin by defining an alternative shadow price variable, μ_t, which is equal to the value of an additional unit of *k* to the benefit stream, valued in period *t* units, i.e., $\mu_t = e^{+rt}\lambda_t$. That is, to get μ_t we have to **in**flate λ_t to convert it from period 0 values to period *t* (current) values.

To help understand what this means, consider an example. Suppose that you have a resource in that in year 30 will have a marginal value of \$1000 and you discount at the rate of 5% per year. What's the value of an increment to the resource, considered from the perspective of year 0, 30 years earlier? Answer: $\$1000/(1.05)^{30}$=\$231. In this case, $\mu_{2050} = \$1000$ while $\lambda_{2050} = \$231$.

The current value Hamiltonian is obtained by inflating (11) to obtain

(12) $H_c = w(k,x) + \mu_t f(k,x,t) = H \cdot e^{rt}$.

There are two differences between (12) and the standard Hamiltonian (11). First, we use μ_t instead of λ_t. Second, the discount factor e^{-rt}, which appears before $w(\cdot)$ in (11), cancels out since we've multiplied the entire function by e^{+rt}.

It is a simple matter to derive the maximum conditions corresponding to H_c and μ instead of *H* and λ.

The first FOC can be rewritten,

$$\frac{\partial H}{\partial x} = e^{-rt}\frac{\partial H_c}{\partial x}$$

so, $\frac{\partial H}{\partial x} = 0$ if and only if $\frac{\partial H_c}{\partial x} = 0$.

Hence the analogous principle holds w.r.t. the control variable, i.e.,

1') $\frac{\partial H_c}{\partial x} = 0$

or, more generally, maximize H_c with respect to *x*.

Now look at the FOC w.r.t. the state variable:
The standard formulation is

$$\frac{\partial H}{\partial k} = -\dot{\lambda}\,.$$

Looking at the LHS of this equation, we see that for the current value Hamiltonian, H_c,

$$\frac{\partial H}{\partial k} = e^{-rt}\,\frac{\partial H_c}{\partial k}$$

and, on the RHS, since $\lambda_t = e^{-rt}\mu_t$

$$-\dot{\lambda} = -\left(-re^{-rt}\mu_t + e^{-rt}\dot{\mu}_t\right) = re^{-rt}\mu_t - e^{-rt}\dot{\mu}_t$$

Putting the LHS and RHS together, we get

$$\begin{aligned}\frac{\partial H}{\partial k} &= -\dot{\lambda}_t \\ e^{-rt}\,\frac{\partial H_c}{\partial k} &= re^{-rt}\mu_t - e^{-rt}\dot{\mu}_t.\end{aligned}$$

Cancelling e^{-rt} gives us the second optimum condition for the current-value specification:

2') $\dfrac{\partial H_c}{\partial k} = r\mu_t - \dot{\mu}_t\,.$

There is no change in the third condition: the state equation must hold.

Finally, the transversality condition might change by a discount factor, but in many cases analogous conditions hold. For example, if the TC is $\lambda_T=0$, and $\lambda_T=\mu_T e^{-rT}$ then it must also hold that $\mu_T=0$. However, for infinite-horizon problems ($T\rightarrow\infty$) if the transversality condition is $\lim_{t\rightarrow\infty}\lambda_t = \lim_{t\rightarrow\infty} e^{-rt}\mu_t = 0$, then a sufficient (though not necessary) condition for the transversality condition to be satisfied is that μ_t is a constant value as $t\rightarrow\infty$, since in this case the discount factor would push $e^{-rt}\mu_t$ to zero.

In conclusion, we can use the current value Hamiltonian in a way very similar to the normal Hamiltonian, but it is important to use the modified optimality conditions.

In summary, we seek to maximize

$W = \int_0^T e^{-rt} w(k,x)dt$ subject to the state equation $\dot{k} = f(k,x,t)$.

We can do this using the current value Hamiltonian,

$H_c = w(k,x) + \mu_t f(k,x,t)$.

where the maximum criteria are:

1') $\dfrac{\partial H_c}{\partial x} = 0$

2') $\dfrac{\partial H_c}{\partial k_t} = r\mu_t - \dot{\mu}_t$

3') $\dfrac{\partial H_c}{\partial \mu} = \dot{k}\,.$

For reasons that we will discuss below, economists tend to use the current-value Hamiltonians for discounted optimization problems, sometimes without even using the "current-value" qualifier.

If you have discounted problem, use H_c.

B. An economic interpretation of the current-value Hamiltonian

As in the standard case, the condition that H_c be maximized over time requires that we strike a balance at every point in time between current and future consequences. The only difference is that now we are considering this tradeoff in terms of the values at future points in time, rather in present value terms. A good way to think about this is by writing it as $\frac{\partial H_c}{\partial x}=\frac{\partial w(k,x)}{\partial x}+\mu_t\frac{\partial f(k,x,t)}{\partial x}=0$ or $\frac{\partial w(k,x)}{\partial x}=-\frac{\partial f(k,x,t)}{\partial x}\mu_t$. The LHS of this expression is the marginal immediate value of a unit of x. On the RHS, $-\mu_t$ is the marginal cost of reducing the capital stock and $\frac{\partial f(k,x,t)}{\partial x}$ tells us how big of an effect on k that a marginal change in x has. So, you can think of the RHS as the value marginal future cost of an increase in the control variable, x. At the optimum, these must be in balance, otherwise the resource manager would be advised to increase or decrease x.

The second condition is a bit trickier, though still easier here than in the present-value specification.[2] Recall that 2' requires

$$\frac{\partial H_c}{\partial k}=\frac{\partial w}{\partial k}+\mu\frac{\partial f}{\partial k}=r\mu-\dot{\mu}$$

which we will rewrite

$$\frac{\partial w}{\partial k}+\mu\frac{\partial f}{\partial k}+\dot{\mu}=r\mu. \tag{13}$$

The three terms of the LHS of this equation can be thought of as a decomposition of the benefits of holding a marginal unit of the capital stock for an instant longer:

- $\frac{\partial w}{\partial k}$ indicates the marginal immediate benefit of the capital stock at time t.
- $\mu\frac{\partial f}{\partial k}$ is the capital stock's marginal value product. That is, $\frac{\partial f}{\partial k}$ tells us how the marginal unit of k contributes to the creation of more k, and this is multiplied by μ the value of that marginal unit of k.
- Finally, $\dot{\mu}$ indicates how the marginal value of the capital is changing over time. If you hold that marginal unit for an instant longer, its value will have changed by $\dot{\mu}$.

The RHS of (13), $r\mu$, can be thought of as the opportunity cost of holding capital. As an example, suppose that our capital good can be easily transformed into dollars, and we

[2] Manseung Han, who took my class in 2002, helped me develop this presentation.

discount at the rate r because it is the market interest rate. Then $r\mu$ is the immediate opportunity cost of holding capital, since we could sell it and earn interest at the rate r.

Hence, along the optimal path we will hold our state variable up to the point where its marginal value is equal to the marginal opportunity cost. That sounds familiar, and very economically reasonable.

Why this makes economic sense is most easily seen when reflecting on the first FOC. Recall that in that case we had marginal current benefit equal to marginal future costs, which makes perfect sense. But what are those marginal future costs along the optimal path? The second FOC helps answer that question.

There are a couple of other ways to look at (13). First, we could rewrite it as

$$\frac{\left(\frac{\partial w}{\partial k}+\mu\frac{\partial f}{\partial k}+\dot{\mu}\right)}{r}=\mu .$$

By dividing the LHS by r, this capitalizes the numerator, indicating how much we could obtain if we received that benefit over an infinite horizon. The numerator then decomposes the marginal value of the capital stock: $\frac{\partial w}{\partial k}$, the value of the marginal unit in terms of its effect on k itself; $\mu\frac{\partial f}{\partial k}$ the value of increments to k due to the marginal unit of k; and the change in the μ itself, $\dot{\mu}$.

Alternatively, we can look at it by rewriting it as

$$\dot{\mu}=\mu r-\frac{\partial w}{\partial k}-\mu\frac{\partial f}{\partial k} .$$

This tells us about how μ changes along the optimal path. First, assuming that $\mu>0$, there is a tendency for it to grow over time at the rate r because of discounting, but it will decline over time if k generates immediate benefits, i.e. if $\frac{\partial w}{\partial k}>0$, and it will tend to decline over time if it gives rise to value benefits through its effect on k itself, i.e. if $\mu\frac{\partial f}{\partial k}>0$. This presentation is closest to the example in section I.D above.

All of these three interpretations are correct, and, in the right context, any of them may help you unravel the economics of a dynamic optimization problem that you might be studying.

C. Summary of the Current-Value Hamiltonian

The current value formulation is very attractive for economic analysis because current values are usually more interesting than discounted values. For example, in a simple economy, the market price of a capital stock will equal the current-value co-state variable. As economists, we are usually more interested in such actual prices than we are in their present value. Hence, the current-value Hamiltonian is more helpful than the present-value variety.

Also, as a practical matter, for analysis it is often the case that the differential equation for μ will be autonomous (independent of *t*) while that for λ will not be. **Hence, the dynamics of a system involving μ can be interpreted using phase-diagram and steady-state analysis; this does not hold for λ.**

One note of caution: we have stated and derived many of the basic results for the present-value formulation (e.g., transversality conditions). When you are using the current-value formulation, you need to be careful to ensure that everything is modified consistently.

III. References

Dorfman, Robert. 1969. An Economic Interpretation of Optimal Control Theory. *American Economic Review* 59(5):817-31.

IV. Readings for next class

Hartwick, John M. "Natural Resources, National Accounting and Economic Depreciation." *Journal of Public Economics* 43(December 1990):291-304. (Available here, must be logged into TAMU account to access).