3. A quick introduction to Optimal Control

AGEC 642 - 2024

I. Why we're not studying calculus of variations.

1) OC is better & much more widely used.
2) Parallels to DP are clearer in OC.
3) I don't know COV, but you can study it in Kamien and Schwartz (1991) Part I.

II. OC problems always[1] contain

- $z_t \Rightarrow$ the (set of) choice variable(s),
- $x_t \Rightarrow$ the (set of) state variable(s),
- $\dot{x}_t = f(t, x_t, z_t) \Rightarrow$ the state equation(s),
- $V = \int_0^T F(t, x_t, z_t)dt \Rightarrow$ an objective function in which $F(\cdot)$ is instantaneous <u>benefit function</u> that captures the rate at which benefits are added to the objective function.[2]
- $x_0 \Rightarrow$ an initial condition for the state variable(s),
- and sometimes explicit in<u>tra</u>temporal constraints, e.g., $g(t,x,z) \leq 0$.

As we saw in the two-period discrete-time model in Lecture 1, OC problems can be solved using the vehicle of the Hamiltonian. In the next lecture we'll see more formally why this holds and then explore the economic intuition behind the Hamiltonian. For now, take my word for it.

The Hamiltonian takes the form: $H = F(t,x_t,z_t)+\lambda_t \cdot f(t, x_t,z_t)$.

The *maximum principle*, due to Pontryagin, states that the following conditions, if satisfied, guarantee a solution to the problem (you should commit these conditions to memory):

1. $\max_z H(t, x_t, z_t, \lambda_t)$ for all $t \in [0,T]$
2. $\dfrac{\partial H}{\partial x_t} = -\dot{\lambda}_t = -\dfrac{\partial \lambda_t}{\partial t}$
3. $\dfrac{\partial H}{\partial \lambda_t} = \dot{x}_t$
4. *Transversality condition* (such as, $\lambda(T) = 0$)

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[1] We consider here only continuous-time problems. In discrete-time problems, the differential equations for the state equations are replaced by difference equations and the objective function has a sum rather than an integral, but it is important to keep track of λ_t, λ_{t-1} and λ_{t+1}.

[2] To reiterate, the benefit function is the instantaneous <u>rate</u> per unit of time that additions to the objective function are made. For example, if the benefit function is $u(z_t)$ then, if $z_t = \bar{z}$, $\int_0^2 u(z_t)\,dt$ would be the value to the decision maker (utility) of keeping z_t fixed at $\bar{z}$ for two periods.

Points to note:

- The maximization condition, 1, is not equivalent to $\partial H/\partial z_t$ =0, since corner solutions are admissible and non-differential and other non-convex programing problems can be considered.
- The maximum criteria include 2 sets of differential equations (2&3), so there's one more set of differential equations than in the original problem.
- Condition 3 is equivalent to saying that the state equation must hold since $\partial H/\partial \lambda_t$ = the RHS of the state equation by the definition of H.
- There are no second-order partial differential equations.

In general, the *transversality condition* is a condition that specifies what happens as we *transverse* to time outside the planning horizon. Above we state $\lambda(T)$=0 as the condition for a problem in which there is no binding constraint on the terminal value of the state variable(s). This makes intuitive sense since λ_t is the marginal value of x_T to the objective function; <u>if</u> you have complete flexibility in choosing x_T, you would want to choose that level so that its marginal value is zero, i.e., λ_T=0. However, λ_T=0 is not always the right transversality condition. We will spend more time discussing the meaning and derivation of transversality conditions in the next lecture.

III. The Solution of an optimal problem (An example from Chiang (1991) with slight notation changes).

$$\max_{z_t} \int_0^T -\left(1+z_t^{\,2}\right)^{1/2} dt \qquad \left(\text{equivalent to } \min_{z_t} \int_0^T \left(1+z_t^{\,2}\right)^{1/2} dt\right)$$

$$s.t. \quad \dot{x}_t = z_t$$

$$\text{and} \quad x_0 = A, \quad x_T \quad \text{free}$$

The Hamiltonian of this problem is

$$H = -\left(1+z_t^{\,2}\right)^{1/2} + \lambda z_t$$

Note that we can use the standard interior solution for the maximization of the Hamiltonian since the benefit function is concave and continuously differentiable. Hence, our maximization equations are

1. $\partial H/\partial z_t = -1/2\left(1+z_t^{\,2}\right)^{-1/2} 2z_t + \lambda_t = 0$

   (if you check the 2nd order conditions you can verify we have a maximum)
2. $\partial H/\partial x_t = 0 = -\dot{\lambda}_t$
3. $\partial H/\partial \lambda_t = z_t = \dot{x}_t$
4. λ_T=0, the transversality of this problem (because of the free value for x_T).

There is no recipe for solving optimal control problems. What usually works best is to work with the easiest parts first and then keep going until you have the solution. This one is quite easy.

a) 2 means that $\partial\lambda_t/\partial t = 0$, i.e., λ_t is constant for all t.

b) Together with 4, this means that λ_t is constant at 0, i.e., λ_t=0 for all t.

c) To find z_t^*, solve 1 after dropping out λ_t and we see that the only way $-1/2\left(1+z_t^2\right)^{-1/2} 2z_t = 0$ is if $z_t^* = 0$.

d) Plug this into the state equation, 3, and we find that x remains constant at A.

Now that was easy, but not very interesting. Let's try something a little more challenging.

IV. A simple consumption problem

$$\max_{z_t} \int_0^1 \ln\left[z_t 4x_t\right] dt$$
$$s.t. \quad \dot{x}_t = 4x_t\left(1-z_t\right)$$
$$and \quad x_0 = 1, \quad x_1 = e^2$$

What would the $\dot{x}=0$ line look like in a phase diagram in x-z space?
What is the transversality condition here
(i.e., what must be true as we transverse beyond the planning horizon)?

The Hamiltonian for this problem is
$H = \ln\left[z_t 4x_t\right] + \lambda_t\left[4x_t\left(1-z_t\right)\right]$

Maximum conditions:

1. $\dfrac{\partial H}{\partial z_t} = \dfrac{1}{z_t} - \lambda_t 4x_t = 0$ (Check 2nd order condition. Do we have a max?)
2. $\dot{\lambda}_t = -\dfrac{\partial H}{\partial x} = -\left[\dfrac{1}{x_t} + \lambda_t 4\left(1-z_t\right)\right]$
3. $\dot{x}_t = \dfrac{\partial H}{\partial \lambda_t} = 4x_t\left(1-z_t\right)$
4. $x_1 = e^2$

Simplifying the first equation yields $\dfrac{1}{\lambda_t 4x_t} = z_t$.

Once you have taken the first order conditions, you can almost always get some economic intuition from the solution (as is required in problem set #2). For example, in this problem we find that current consumption is inversely related to the product of the state and costate variables. Does this make intuitive sense?

Substituting for z_t in 2

$$\dot{\lambda}_t = -\frac{1}{x_t} - \lambda_t 4\left(1 - \frac{1}{\lambda_t 4x_t}\right)$$

$$\dot{\lambda}_t = -\frac{1}{x_t} - \left(\lambda_t 4 - \frac{1}{x_t}\right)$$

$$\dot{\lambda}_t = -\lambda_t 4$$

Can you solve the differential equation to obtain λ_t as a function of t? Hint: $\dot{\lambda}_t / \lambda_t = -4$

Now, substituting for z_t in the state equation, we obtain

$$\dot{x}_t = 4x_t\left(1 - \frac{1}{\lambda_t 4x_t}\right)$$

So, our three simplified equation are

5. $\dfrac{1}{\lambda_t 4x_t} = z_t$
6. $\dot{\lambda}_t = -\lambda_t 4$
7. $\dot{x}_t = 4x_t - \dfrac{1}{\lambda_t}$

Is there an equilibrium where both $\dot{\lambda}$ and $\dot{x}$ equal zero?

Notice that 6 involves one variable, 7 involves two variables and 5 involves three variables. This suggests an order in which we might want to solve the problem – start with 6.

The differential equation in 6 can be expressed $\dfrac{\dot{\lambda}_t}{\lambda_t} = k$, that is λ grows at a constant rate. Such specifications always have a solution $\lambda_t = \lambda_0 e^{k \cdot t}$. Hence 6 implies

8. $\lambda_t = \lambda_0 e^{-4t}$

(where λ_0 serves at the constant of integration, which is the value of λ when t=0).

[⇒ check: if $\lambda_t = \lambda_0 e^{-4t}$, then $\partial\lambda/\partial t = \dot{\lambda}_t = -4\lambda_0 e^{-4t} = -4\lambda_t$ ✓]

This solution for λ_t can then be substituted into 7 to get

$$\dot{x}_t = 4x_t - \frac{e^{4t}}{\lambda_0},$$

which is a simple FODE. Building on the way we solve linear FODE's in Lecture 2:

$$e^{-4t}\left(\dot{x}_t - 4x_t\right) = -\frac{1}{\lambda_0}$$

$$e^{-4t}\dot{x}_t - 4e^{-4t}x_t = -\frac{1}{\lambda_0}$$

We can integrate both sides of this equation over t

LHS: $\int\left[e^{-4t}\dot{x}_t - 4e^{-4t}x_t\right]dt = x_t e^{-4t} + A_1$

RHS: $\int -\frac{1}{\lambda_0}dt = -\frac{t}{\lambda_0} + A_2$

so

$$x_t e^{-4t} + A_1 = -\frac{t}{\lambda_0} + A_2$$

or

$$e^{-4t}x_t = -\frac{t}{\lambda_0} + A$$

or

9. $x_t = -\frac{te^{4t}}{\lambda_0} + Ae^{4t}$

where A is an unknown constant.

We are close to the solution. However, <u>we are not finished until the values for all constants of integration have been found</u>. To do this we use the initial and terminal conditions (a.k.a. transversality condition).

Using 9 and substituting in, x_0=1, and t=0, yields

$$1 = -\frac{0\cdot e^{4\bullet 0}}{\lambda_0} + A\cdot e^{4\bullet 0} = A;$$

so, A=1.

Now use the terminal condition (4): $x_1 = e^2$, and substituting that into 9 when t=1,

$$e^2 = -\frac{1\cdot e^{4\cdot 1}}{\lambda_0} + e^{4\cdot 1}$$

$$\frac{e^4}{\lambda_0} = e^4 - e^2$$

$$\frac{e^4}{e^4 - e^2} = \lambda_0$$

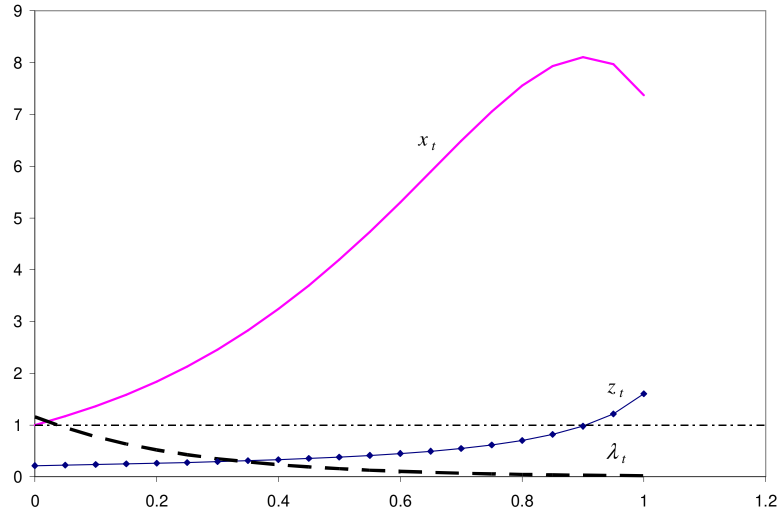
$$\lambda_0 \approx 1.156$$

Now plug the values for A and λ_0 into 8 and 9 to get the complete time line for λ and x:

$\lambda_t = (1.156)e^{-4t}$ and $x_t = e^{4t}-0.865te^{4t}$. These can then be substituted into 5 to get

$$z_t = \frac{1}{4.624 - 4t}$$

Hence, the solution to the problem can be graphed as follows.



Are these curves consistent with economic intuition?

V. An infinite horizon resource management problem

Let's look at one more problem, the case of a renewable resource, a fishery, in which the stock of fish in a lake, x_t, changes continuously over time. We assume that the natural rate of growth in the stock of the fish is $ax_t - b(x_t)^2$, a,b>0, but the rate of change in the stock is also affected by the rate at which fish are being harvested, z_t. So, the equation of motion is

$$\dot{x}_t = ax_t - b(x_t)^2 - z_t.$$

Yield is sold for price P and each unit of extraction costs c, so society's utility comes from harvests if a function of profits, $(P-c)z_t$. We will assume that utility is equal to the log of profits, and we will define $(P-c)=p$ so that the rate at which societal welfare is generated is $\ln(pz_t)$. The goal is to maximize the discounted present value of its utility over an infinite horizon, discounting at the rate r.

A formal statement of the planner's problem, therefore is:

$$\max_{z_t} \int_0^\infty e^{-rt} \ln(p \cdot z_t) \quad s.t.$$

$$\dot{x}_t = ax_t - bx_t^2 - z_t$$

$$x_t \geq 0$$

We solve this problem using a Hamiltonian:

$$H = e^{-rt} \ln(p \cdot z_t) + \lambda_t \left(ax_t - b(x_t)^2 - z_t\right),$$

yielding the first-order conditions:

1. $\dfrac{e^{-rt}}{z_t} = \lambda_t$
2. $\lambda_t\left(a - 2bx_t\right) = -\dot{\lambda}_t$
3. $\dot{x}_t = \left(ax_t - b\left(x_t\right)^2 - z_t\right)$
4. $\lim\limits_{t\to\infty} \lambda_t = 0.$

A common approach used in infinite-horizon problems is to look at the phase diagram to explore the dynamics of the system. As discussed in Lecture 2, a phase diagram presents the relationships between two autonomous differential equations, but we have three variables. The state equation provides the first, specifying the dynamic relationship between x_t and z_t:

$$\dot{x}_t = \left(ax_t - b\left(x_t\right)^2 - z_t\right).$$

The second differential equation comes from 2:

$$-\frac{\dot{\lambda}_t}{\lambda_t} = \left(a - 2bx_t\right).$$

In order to develop the phase diagram, we need to choose whether we want to do it in x-z space of x-λ space. We can choose either since from equation 1 there is a 1:1 relationship between λ and z at any point in time, t. Let's use x-z.[3]

$$\lambda_t = \frac{e^{-rt}}{z_t}$$

$$\ln\left(\lambda_t\right) = -rt - \ln\left(z_t\right)$$

$$\frac{\dot{\lambda}_t}{\lambda_t} = -r - \frac{\dot{z}_t}{z_t}$$

Hence, we can rewrite 2 as

$$r + \frac{\dot{z}_t}{z_t} = \left(a - 2bx_t\right) \Rightarrow \dot{z}_t = \left(a - r - 2bx_t\right)z_t.$$

The two equations for our phase diagram, therefore, are

$$\dot{z}_t = \left(a - r - 2bx_t\right)z_t$$

$$\dot{z}_t \geq 0 \Rightarrow \left(a - r - 2bx_t\right)z_t \geq 0$$

since, $z_t > 0$ by the $\ln(\cdot)$ function

$$\Rightarrow a - r - 2bx_t \geq 0$$

$$\Rightarrow \frac{a-r}{2b} \geq x_t$$

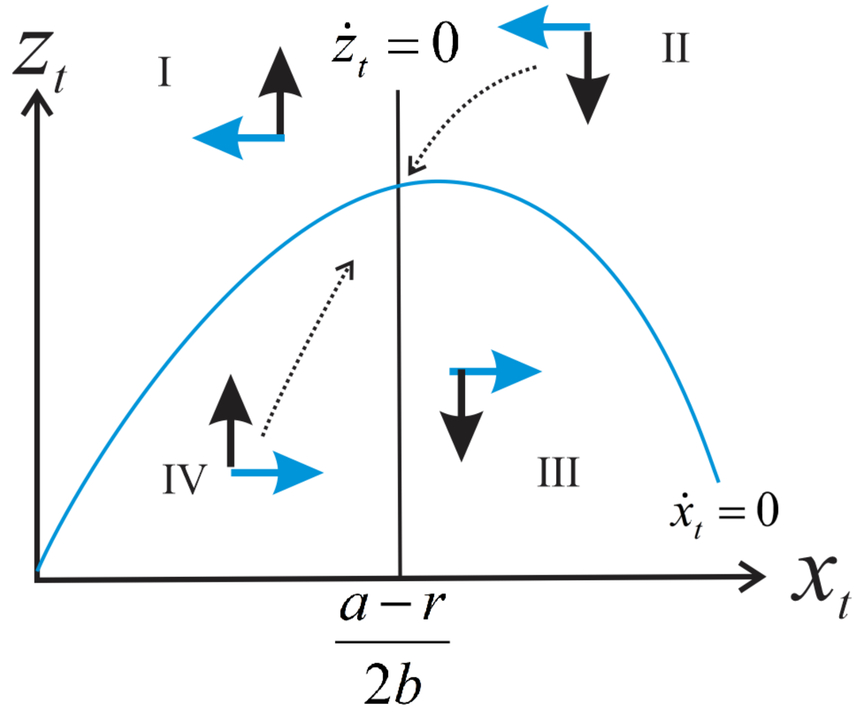
and

$$\dot{x}_t = \left(ax_t - b\left(x_t\right)^2 - z_t\right)$$

$$\dot{x}_t \geq 0 \Rightarrow ax_t - bx_t^2 - z_t \geq 0$$

$$\Rightarrow ax_t - bx_t^2 \geq z_t$$

[3] A phase diagram in x-λ space is not possible. We can see this by using FOC 1 to substitute λ for z in FOC 3 to obtain $\dot{x}_t = \left(ax_t - b\left(x_t\right)^2 - e^{-rt}/\lambda_t\right)$, which is a function of t. So, the differential equation is not autonomous.



It is clear from the diagram that we have a saddle path equilibrium with paths in quadrants II and IV. However, it is important to remember that all of the dynamics presented in the phase diagram are consistent with the first order conditions 1 – 3. We can now use the constraint $x_t \geq 0$ and the transversality condition to show that <u>only</u> points that are on the saddle paths are fully optimal.

In quadrant I of the phase diagram, all paths lead to decreasing values of x and increasing values of z. Along such paths $\dot{x}_t = ax_t - b(x_t)^2 - z_t$ is negative and growing in absolute value; eventually x would have to become negative. But this violates the constraint on x; so, such paths are not admissible in the optimum.

In quadrant III, harvests are declining and the stock is increasing. Eventually this will lead to a point where x reaches the biological steady state where natural growth is zero so harvests, z_t must also be zero. This will occur in finite time. But that means at such a point $\lambda_t = \infty$, which violates the transversality condition. Hence, as with quadrant I, no point in quadrant III is consistent with the optimum.

Finally, we can also rule out any point in quadrants II or IV that are not on the saddle path because if the path does not lead to the equilibrium, it will eventually cross over to quadrant I or III or reach one of the axes; no such paths are consistent with the optimum Hence, only points on the separatrices are solutions to the planner's optimization problem.

What would the phase diagram in x-λ space look like? How about z-λ?

This problem also gives us an opportunity to consider how the value of the discount rate, r, affects the optimal behavior. As can be easily seen from the phase diagram, r affects the isocline for z, but has not effect on the isocline for x. Hence, as r increases, the equilibrium moves to the left with a lower equilibrium value for both z and x.

How does this happen? Suppose the initial stock, x_0, is greater than $\frac{a}{2b}$, the maximum of the x isocline. Until the equilibrium is reached, harvests will exceed growth. As r increases, harvests will continue to exceed growth longer, pushing the eventual equilibrium for x to a lower level, where the equilibrium level of harvests is also lower.

Why does this happen? Because as r increases, the planner puts more weight on the short term, and discounts the period out to infinity when the stock is at the equilibrium.

VI. References

Chiang, Alpha C. 1991. *Elements of Dynamic Optimization*. McGraw Hill

VII. Readings for next class

Chiang pp. 181-184
Léonard & van Long Chapter 7