2. The basics of differential equations[1]

AGEC 642 – Fall 2024

I. What is a differential equation?

A *differential equation* is an equation that involves a derivative of a function. In our applications, typically the equation will define a function that is equal to the derivative with respect to time, e.g.,

$$\frac{\partial x(t)}{\partial t} = f(x,z,t)$$

The LHS of this equation will frequently be written $\dot{x}(t)$, $\dot{x}_t$ or just $\dot{x}$.

Note that differential equations are used when time is measured continuously. The term *difference equation* is used for the discrete-time analog. We can see the relationship between a difference equation by considering a situation in which $x_{t+1} = x_t + f(x_t, z_t)$, that is, $f(x_t, z_t)$ is a function that tells us how much x changes from one period to the next. Now consider a time step of Δ<1. We can now write

$$x_{t+\Delta} = x_t + \Delta \cdot f(x_t, z_t), \qquad (1)$$

i.e., $f(x_t, z_t)$ is the rate of change in x per period, but since Δ is not a full time step, the change from t to t+Δ is only $\Delta \cdot f(x_t, z_t)$. Subtracting x_t from both sides of and dividing by Δ we obtain

$$\frac{x_{t+\Delta} - x_t}{\Delta} = f(x_t, z_t).$$

Since Δ is a change in time, we can now take the limit as Δ approaches zero to obtain the partial derivative of x with respect to time:

$$\lim_{\Delta \to 0} \frac{x_{t+\Delta} - x_t}{\Delta} = \frac{\partial x_t}{\partial t} = \dot{x}_t = f(x_t, z_t).$$

Since discrete-time processes are often more intuitive, it will sometimes be helpful to replicate this derivation of the partial derivative.

Part of the difficulty (i.e., hassle) of optimal control is that the first order conditions yield differential equations, which we have to integrate to obtain a closed form solution to the problem. Hence, to solve optimal control problems we have to understand differential equations and be able to solve them.

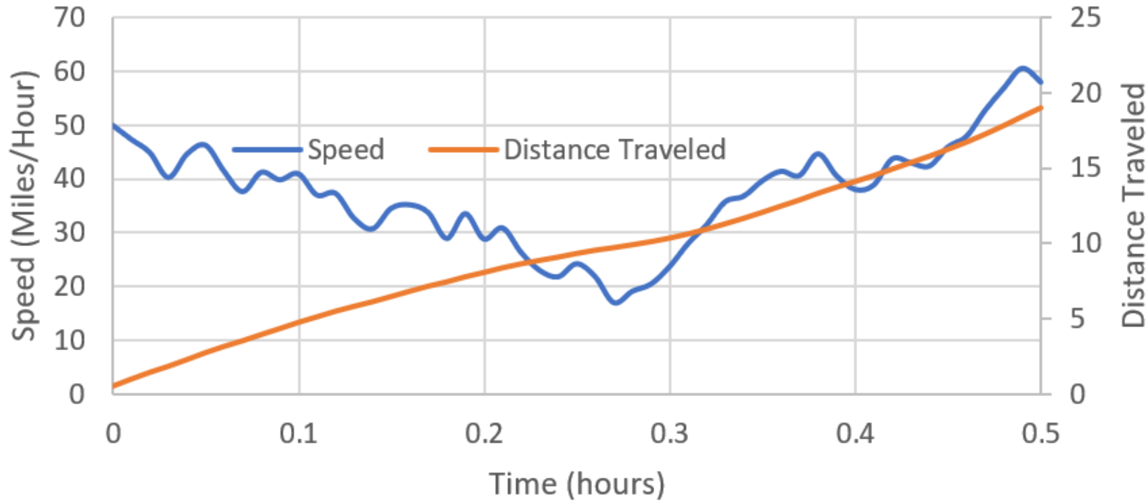
Notation: We will frequently write $x(t)$, x_t or just x, all meaning the same thing. Mostly we will use x_t for completeness and concision. The correct meaning should be understandable in context; if not, ask.

Example 1: Suppose x_t is the distance traveled up to time t. The rate of change in distance with respect to time is $\dot{x}_t$, and the units of this measurements are distance per unit of time, such as miles per hour or centimeters per second. If you can choose your speed at any

[1] These notes are based primarily on chapter 2 of Léonard and Van Long.

instant, then your speed is a choice variable, which we will refer to as z_t. Hence, the differential equation describing this relationship is $\dot{x}_t = z_t$ and the distance you've traveled after k hours is $x_k - x_0 = \int_0^k \dot{x}_t dt = \int_0^k z_t dt$. If your speed is constant, i.e., z_t=z, then we can easily solve the differential equation $x_k = \int_0^k z_t dt + x_0 = k \cdot z + x_0$. If x_0=0, then x_k=$k{\cdot}z$. So, if k=0.5 hours and z=50 miles per hour, then x_k=25 miles. Another option would be to travel 50 miles per hour for 15 minutes and 70 miles per hour for 15 minutes, which would give us $x_{0.5} = \int_0^{0.5} z_t dt = \int_0^{0.25} 50 dt + \int_{0.25}^{0.5} 70 dt = 30$.

More generally, your speed can fluctuate continuously over the period, in which case z_t and x_t might look something like this.



It is important to recognize that **when integrating over time, the integrand** (i.e., what is inside the integral, z in the case above) **is <u>always a rate</u> of change per unit of time as defined by how we measure *t*.** So, for example, if we changed our units from miles per hour to miles per minute, then the value of z would change to z_t'=z_t/60, and we would integrate not from 0 to k but from 0 to 60×k. In our applications, instead of putting speed in the integral, we will often put a utility function. Hence, <u>the correct interpretation of the utility function is also a rate, **the rate at which utility is being created per period**</u>.[2]

Example 2: Suppose an investment of x_0 dollars at time t=0 is put in an account so that the balance grows continuously at the rate of interest r, i.e., $x_t = x_0 e^{rt}$.

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[2] An interesting side note: The psychological literature has found evidence that people do *<u>not</u>* seek to maximize the integral of utility over time. The best-selling book, *Thinking Fast and Slow* by Daniel Kahneman (winner of the Nobel Prize in Economics) talks a lot about how people seem to pay more attention the peak and end points in an experience. If this is right, then the discounted utility model of economists may be a very poor descriptive model of behavior. Nonetheless, it is what economists normally use and there are good economic and theoretical reasons why discounted utility is a "rational" objective.

In this case the differential equation is

$\dot{x}_t = \frac{\partial x_t}{dt} = \frac{\partial x_0 e^{rt}}{dt} = r x_0 e^{rt} = r x_t$, which implies that $\frac{\dot{x}_t}{x_t} = r$.

Often, however, we are given the differential equation itself, in this case

$$\dot{x} = rx ,$$

and we need to obtain x_t, and we do that by integrating.

An easy way to integrate $\dot{x} = rx$ is to use the fact that we can write this as $\frac{\dot{x}}{x} = r$.

We want to solve for x as a function of other stuff, so we start by integrating both sides,

$$\int_t \frac{\dot{x}}{x} dt = \int_t r dt \, . \tag{2}$$

From the chain rule we know that $\frac{\partial \ln(x_t)}{\partial t} = \frac{1}{x}\frac{\partial x_t}{\partial t} = \frac{\dot{x}}{x}$, so the LHS can be easily integrated $\int_t \frac{\dot{x}}{x} dt = \ln(x_t)$ plus a constant of integration. The integral of r on the RHS is just $r \cdot t$. So (2) can be rewritten

$$\ln(x_t) = rt + K ,$$

where K is the constant of integration. Note that since the constant of integration is unknown on both the left and the right, they can be combined to obtain a single constant, K.

Taking the *exp* of both sides, we obtain $x_t = e^{rt+K} = e^K e^{rt} = Ae^{rt}$, where $A = e^K$ has an unknown value.

If we know the value of x at some point in time, e.g., if $x_0 = \bar{x}$ (i.e., we have $\bar{x}$ dollars in the bank at time t=0) then, substituting for t=0, we can solve for A,

$$\bar{x} = Ae^{r \cdot 0} = A .$$

Hence, the *specific solution* is $x_t = \bar{x}e^{rt}$.

This example demonstrates the standard process of solving differential equations: first find the *general solution*, then use prior information about value(s) at point(s) in time to get the specific solution. (What would the answer look like if instead of knowing, x_0 we knew that $x_s = K$ for s>0?)

II. Normal differential equations

The equation $\dot{x} = rx$ is a **first-order differential equation** (FODE), *first-order* in the sense that $\dot{x}_t$ is the <u>first</u> derivative of x_t with respect to t. In general, an ordinary m^{th} order differential equation is an equation of the form

$$\frac{\partial^m x}{\partial t^m} \equiv x^{(m)} = g\left(x^{(m-1)}, x^{(m-2)}, \ldots, x^{(2)}, \dot{x}, x, t\right).$$

For example, a FODE would have $\dot{x}_t$ on the LHS and x_t and t on the RHS. A second order differential equation would have $\ddot{x} = \partial^2 x_t / \partial t^2$ on the LHS could have and $\dot{x}_t$, x_t and t on the RHS.
If you have the m^{th} order equation, in principle you recover an equation for x_t as a function of t by integrating m times. Solving the m^{th} order differential equation will lead to an $m{-}1^{th}$ order differential equation, and solving that will lead to an $m{-}2^{th}$ order differential equation, and so on.

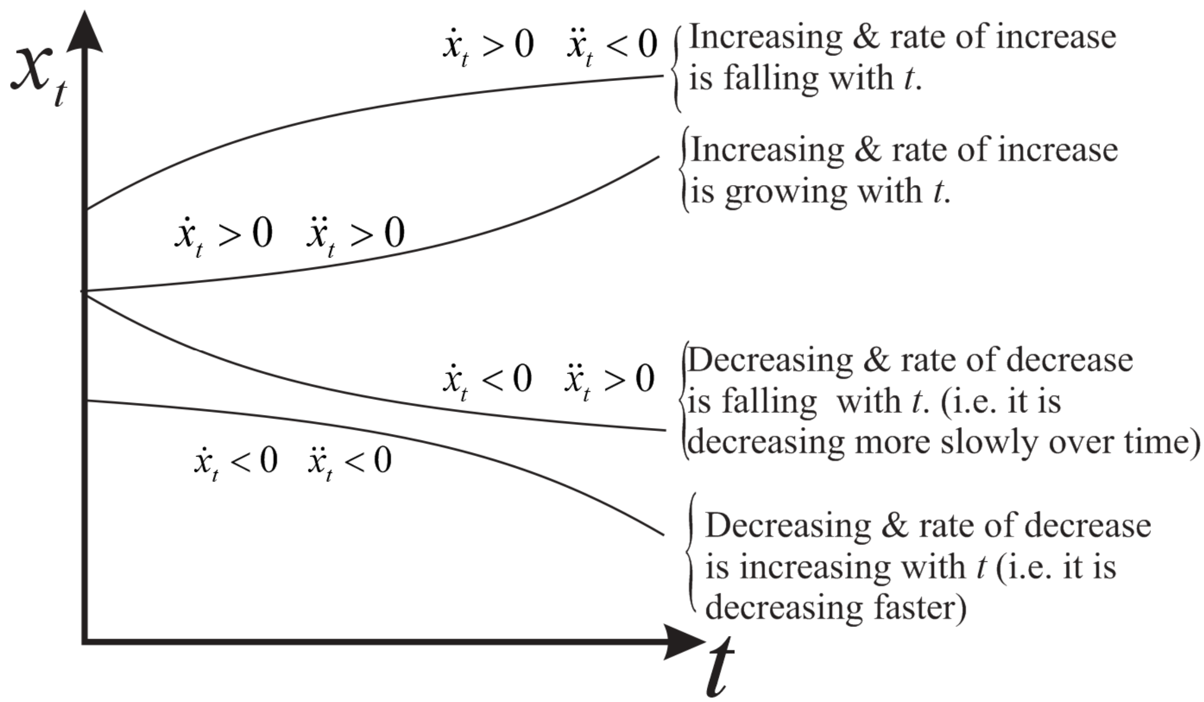
Each time you integrate, however, you end up with a new constant of integration like K above. Hence, to reach a particular solution to an m^{th} order equations you will need m "boundary conditions" (i.e., the exact value of x, $\dot{x}$, $\ddot{x}$, or some other derivative up to $x^{(m-1)}$ at some t. Despite the name, these values do not need to be known at any boundary; the value at any point in time will do, though usually we have the value at the beginning or end of the time horizon.

III. The <u>units</u> and meaning of a differential equation

At the risk of being repetitive, let's look again at the real-world meaning of differential equations. If x_t is a state variable, then it must have <u>units</u> in which it is measured. For example, x_t might be tons, gallons, dollars, miles traveled, etc. Similarly, the <u>units</u> for the time step, ∂t, must also be clearly defined; it could be seconds, hours, years, etc.

For example, if you measure time in hours, and x_t is a distance measured in miles, then the units for $\partial x_t / \partial t = \dot{x}_t$ is a value in miles-per-hour. Note that this is <u>**not**</u> the number of miles that are traveled in one hour, because the vehicle's speed could change continuously over the hour, and you may not drive for a full hour. Rather, it is an *instantaneous* measure of speed, and $\dot{x}_t$ is equal to the distance that the vehicle would travel in one hour ***if*** the speed were held constant for that hour. The integral $\int_0^1 \dot{x}_t \partial t$ would be the number of miles actually traveled in one hour. There is also intuitive meaning in the second derivative, $\partial^2 x_t / \partial t^2 = \partial \dot{x}_t / \partial t = \ddot{x}_t$; this is the rate of change in $\dot{x}_t$, i.e., the rate of acceleration. Note that if we measured the time step t is seconds or years the value of $\dot{x}_t$ would change accordingly, even though the speed of the vehicle has not changed.

It is important to have a clear understanding of what $x_t, \dot{x}_t$ and $\ddot{x}_t$ mean intuitively. If x_t is a positive number, the quantity of grain in your inventory for example, and $\dot{x}_t$ >0, then your inventory is growing and if $\dot{x}_t$ <0, then your inventory is falling. If $\dot{x}_t$ >0 but $\ddot{x}_t < 0$, then the inventory is still growing, but the speed at which it is growing is slowing down. If $\dot{x}_t$ <0 but $\ddot{x}_t > 0$, then the inventory is falling , but the speed at which it is falling is slowing down. The figure below should be helpful in thinking through these meanings.



IV. Equilibrium

An equilibrium in a dynamic system is a point at which all the variables do not change over time. (Students often confuse equilibrium with optimum; be sure you understand the difference.)

If $\dot{x}_t = g(x_t)$ and $g(\bar{x}) = 0$ then $\bar{x}$ is an equilibrium value of x.

- What's the equilibrium for the FODE $\dot{x} = ax + b$?
- What's the equilibrium for the FODE $\dot{x} = rx$?
- What's the equilibrium for the FODE $\dot{x} = ax^2 + bx + c$?
- What's the equilibrium if $\dot{x}_1 = 3x_2 - x_1$ and $\dot{x}_2 = 4 + x_1$?

V. Linear first-order differential equations (FODE)

Linear first-order differential equations are the simplest form of differential equations, and the type we will be using most often. A linear FODE is an equation of the form
$\dot{x}_t = ax_t + b$.

It is instructive to walk through one way to solve such equations. First, we multiply both sides by e^{-at} and reorganize:
$e^{-at}\dot{x}_t - e^{-at}ax_t = e^{-at}b$.

Notice that the LHS of this equation is the time-derivative of $e^{-at}x_t$ using the product rule $\left(\partial e^{-at}x_t/\partial t = -ae^{-at}x_t + e^{-at}\dot{x}_t\right)$, so $\int\left(e^{-at}\dot{x}_t - e^{-at}ax_t\right)dt = e^{-at}x_t + C_1$, where C_1 is a constant of integration. The RHS can easily be integrated, $\int e^{-at}b\,dt = e^{-at}\dfrac{b}{-a} + C_2$.

Noting that only one constant of integration, C, can be identified, integrating the LHS and the RHS we obtain $e^{-at}x_t = \frac{1}{-a}e^{-at}\cdot b + C$, or , canceling e^{-at},

$$x_t = -\frac{b}{a} + Ce^{at}. \tag{3}$$

It is always a good idea to check your integration. In this case, taking the derivative of (3) with respect to t we obtain $\frac{\partial x_t}{\partial t} = \dot{x}_t = aCe^{at}$. But, using (3) we know that $Ce^{at} = x_t + \frac{b}{a}$, so we can write the derivative $\dot{x}_t = a\left[x_t + \frac{b}{a}\right]$, or $\dot{x}_t = ax_t + b$. ✓

Solving another relatively simple FODE:

- Suppose we have a FODE that can be written in the form
  $\dot{x}_t = h(x_t)g(t)$.
  If we define $f(x_t) = 1/h(x_t)$, then our FODE can be rewritten $f(x_t)\dot{x}_t = g(t)$.
  Then both sides can be integrated w.r.t. t to obtain
  $\int f(x_t)\frac{\partial x}{\cancel{\partial t}}\cancel{dt} = \int g(t)\,dt$
  $\int f(x_t)\,dx_t = \int g(t)\,dt.$
  So, the LHS involves integration of x_t while the RHS involves integration over t.

If you have to deal with more complicated differential equations, there are a number of good computer programs that can help. Given the sophisticated software available today (e.g., Matlab, Maple, & Mathematica), solving complicated differential equations entirely by hand is almost like doing OLS with a hand calculator. We will go over the use of such software in the computer lab (and see the Matlab tutorial that accompanies these notes).

VI. Autonomous ODEs

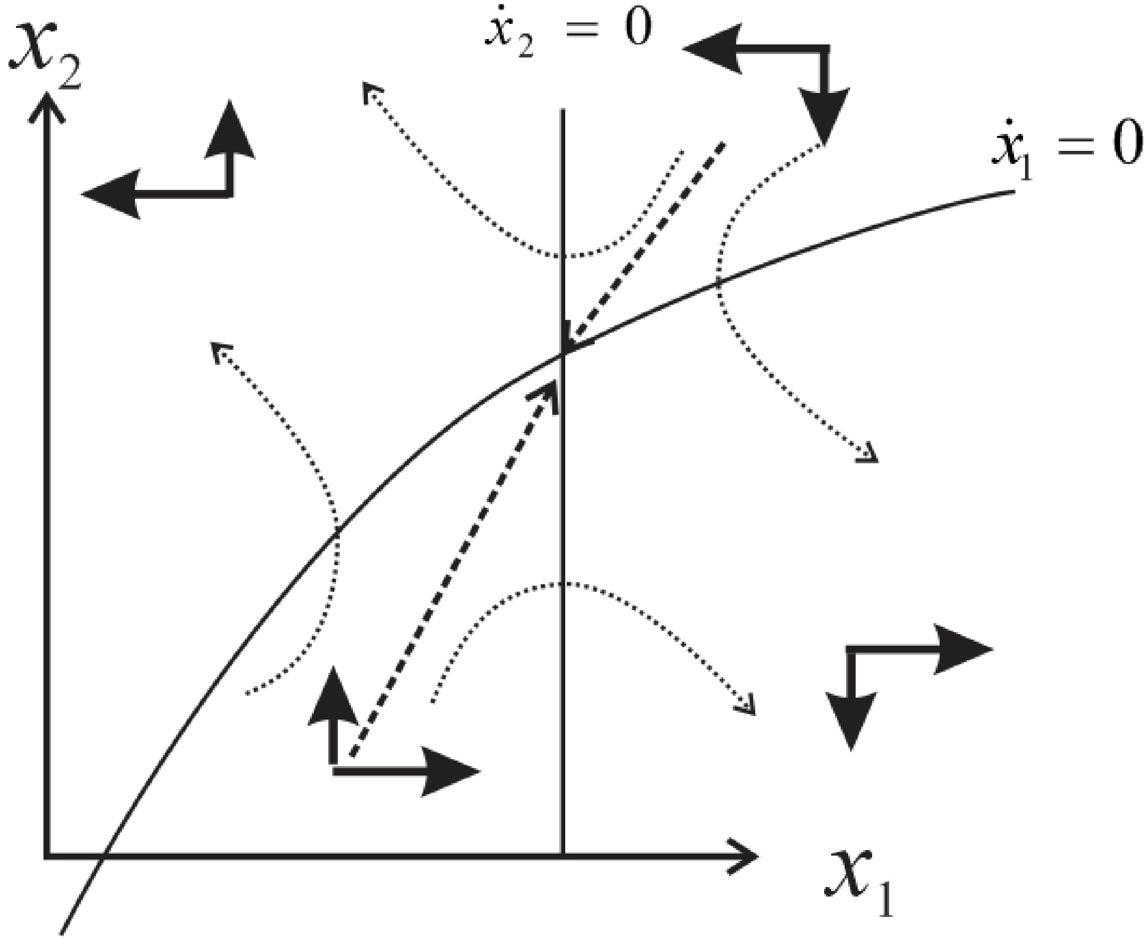
A differential equation is said to be <u>autonomous</u> if it does not depend on t. More formally, according to Weisstein's MathWorld, "For an autonomous ODE, the solution is independent of the time at which the initial conditions are applied." In economics, we frequently seek to specify our problems to be autonomous since we typically feel that economic changes are a function of the state of the system and the choices made (and perhaps random shocks); we usually do not think that the calendar date itself is driving economic outcomes. For example, think of climate change. While we may use a time trend in the analysis, this is a proxy for the accumulation of greenhouse gases in the atmosphere and the greenhouse gases would be a state variable.

As an example of a differential equation that is **not** autonomous, consider $\dot{x}_t = at + bx_t$; the rate of change in x depends not only on the value of x but the time, t. On the other hand, the function $\dot{x}_t = y_t + bx_t$ **is** autonomous, as long as y_t is not itself a function of time.

VII.Systems of differential equations and phase diagrams

Frequently in OC we have to deal with more than one differential equation at a time. In the simplest OC problems, for example, we have a differential equation for the state variable, $\dot{x}_t$, and another for the co-state variable, $\dot{\lambda}_t$. In other cases, we might have two state variables, e.g., two interdependent fish stocks or the market shares of two competing firms.

Without solving explicitly for the entire time path of the two variables, we can learn quite a lot about the nature of a two-variable system using what is called a *phase diagram*. A phase diagram presents the equilibria, stability and dynamic evolution of a system. Phase diagrams are appropriate only if you have two autonomous differential equations. An example of a phase diagram is shown below. We will discuss the steps to develop a phase diagram toward the end of these notes.



The type of system portrayed here is known as a saddle point or saddle path and is frequently encountered in economic models. The solid lines are called isoclines, indicating that along these lines there is no direct pressure on one of the variables to change, $\partial x_i / \partial t = 0$. The equilibrium occurs where the isoclines cross – where both variables do not change. The dashed lines heading toward the equilibrium are called *separatrices* since they separate the space in that no trajectory ever crosses these lines; if a path reaches a separatrix it never leaves it. The dotted lines in this figure are representative trajectories that are not on the separatrices. Note that when a trajectory crosses an isocline its slope is consistent with the isocline. For example, the bottom right trajectory is horizontal at the point where it crosses the $\dot{x}_2 = 0$ isocline because at that point x_2 is neither increasing nor decreasing.[3] In a saddle point system, only points on the separatrices will lead to the equilibrium; if a starting point is not on one of these lines it will permanently diverge from the equilibrium.

VIII. Homogeneous and non-homogeneous systems

Consider first a system of linear differential equations

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 - b_1$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 - b_2$$

which can be written

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

or, using matrix notation,

$$\dot{\mathbf{x}} = \mathbf{Ax} - \mathbf{b}. \tag{4}$$

Using a phase diagram, the equilibrium of this system could easily be identified as the point where $\dot{\mathbf{x}} = 0$ or $\mathbf{Ax} = \mathbf{b}$. We can, therefore, find the equilibrium values, $\bar{\mathbf{x}}$**,** by inversion, $\bar{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{b}$.

It is also frequently interesting to know how variables behave around the equilibrium. For example, do x_1 and x_2 tend toward the equilibrium, or away from it? It turns out that except in a special case (see L&VL p. 101), the dynamics of the system in 4 will be identical to the dynamics of the related *homogeneous* system in which the **b** is dropped:

$$\dot{\mathbf{x}} = \mathbf{Ax}. \tag{5}$$

The only difference between the system defined by 5 and the system defined by 4 is that the equilibrium relocated from $\mathbf{A}^{-1}\mathbf{b}$ to the origin. A system of differential equations in which the equilibrium is at the origin is called *homogeneous*.

IX. Analysis of the nature of the equilibria in systems of differential equations

The nature of the equilibrium of a system of differential equation can be determined by looking at the Eigen values of the system.

You have probably seen Eigen values before. You'll recall that the first step in identifying the Eigen values is to "guess" the solution to the homogenous differential equation system

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[3] When talking about a single variable, x_t above, I have tried to be careful to retain the *t* subscript to remind us that x_t, $\dot{x}_t$, $\ddot{x}_t$ etc., all change over time. When talking about systems with multiple state variables, the time subscript is suppressed, but do not forget that to be complete we would write x_{1t} and x_{2t} to emphasize that these too will also be changing over time.

takes a form analogous to the scalar case discussed above.[4] That is, we could "guess" that the solution will look something like

$$\mathbf{x} = \mathbf{a}e^{\lambda t}, \tag{6}$$

where **a** is a vector of constants, not all zero. Taking the time derivative of this function we obtain,

$$\dot{\mathbf{x}} = \lambda \mathbf{a}e^{\lambda t}. \tag{7}$$

Setting the RHSs of 5 and 7 equal, we get

$\lambda \mathbf{a}e^{\lambda t} = \mathbf{Ax} = \mathbf{Aa}e^{\lambda t}$.

Canceling $e^{\lambda t}$, we get

$\lambda \mathbf{a} = \mathbf{Aa}$ or

$[\mathbf{A} - \lambda \mathbf{I}]\mathbf{a} = 0$.

or

$$\begin{bmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

For nontrivial solutions, i.e., $\mathbf{a} \neq 0$, this requires that $[\mathbf{A} - \lambda \mathbf{I}]$ be singular, i.e., $|\mathbf{A} - \lambda \mathbf{I}| = \mathbf{0}$.[5] A value λ that satisfies this is called an Eigen value or a characteristic root.

For a 2×2 matrix, solving the equation where $|\mathbf{A} - \lambda \mathbf{I}|$ is equal to $(a_{11} - \lambda_1)(a_{22} - \lambda_2) - a_{21}a_{12} = 0$. This is a nonlinear equation, quadratic if $\lambda_1 = \lambda_2 = \lambda$. For any real 2×2 matrix **A**, the Eigen values form part of a matrix **B** such that there exists a real matrix **T** such that $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{B}$. **B** can take one of four forms,

$$(a)\ \mathbf{B} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad (b)\ \mathbf{B} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$(c)\ \mathbf{B} = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}, \quad (d)\ \mathbf{B} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

where λ_1 and λ_2 are distinct real roots, λ is a double root and $\alpha \pm i\beta$ are conjugate complex roots where $i = \sqrt{-1}$. Depending on the roots, the stability of the system falls into six categories (see Léonard and van Long p. 98) and these determine whether the system is stable (converging towards the equilibrium) or unstable.

Some intuition about stability of the homogenous system can be found by looking at 6. If λ is negative, then as *t* increases **x** will approach zero, which is the equilibrium of the homogeneous system. If λ is positive, then **x** will grow as *t* increases, moving away from the equilibrium. If $\lambda = [\lambda_1, \lambda_2]$, and one is greater than zero and the other is less than zero, things are likely to be more complicated.

[4] This section, like almost all of this lecture, is based very closely on chapter 2 of Leonard & Van Long. A student in the past has questioned this section. If you too question this, I would welcome a clarification and/or correction of the derivation.

[5] The notation $|\mathbf{A} - \lambda \mathbf{I}|$ refers to the determinant of the matrix $\mathbf{A} - \lambda \mathbf{I}$, where **I** is an identity matrix.

You can calculate these Eigen values by hand by solving the equation $|\mathbf{A}-\lambda\mathbf{I}|=\mathbf{0}$, or you can use a software package to solve for the Eigen values. The following sequence of Matlab commands will calculate Eigen values of a 2×2 matrix.

```
EDU>> syms A B a b c d
EDU>> A=[a,b;c,d]
EDU>> B=eig(A)
```

$$\Rightarrow A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

With the result being

```
B =
[ 1/2*a+1/2*d+1/2*(a^2-2*a*d+d^2+4*b*c)^(1/2)]
[ 1/2*a+1/2*d-1/2*(a^2-2*a*d+d^2+4*b*c)^(1/2)]
```

Some simple rules that establish stability of a system of differential equations

Fortunately, there are some simple rules that are very helpful in quickly analyzing the dynamics of many systems like 5. As noted by L&VL (p. 100):

i. Such a system has a stable equilibrium if and only if its characteristic roots have negative real parts.
ii. A saddle point occurs if and only if the determinant of **A** is negative.
iii. A sufficient condition for instability is that the trace of **A**>0.

Note that conditions ii and iii can occur simultaneously. For all but case d above, the determinant of **A**, $|\mathbf{A}|=\lambda_1\cdot\lambda_2$ and tr $\mathbf{A}=\lambda_1+\lambda_2$, so conditions ii and iii can be evaluated with the roots, or with the original **A** matrix.

Examples: Consider the following possible **A** matrices for a 2-variable system. Using conditions ii and iii, what do we know about each system?

$$A=\begin{bmatrix} -4 & 1 \\ 0 & 3 \end{bmatrix}\quad A_2=\begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}\quad A_3=\begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}\quad A_3=\begin{bmatrix} 1 & 2 \\ -5 & -2 \end{bmatrix}$$

saddle:	☐	☐	☐	☐
stable:	☐	☐	☐	☐

Nonlinear systems

Of course, the analysis that we have developed here starting with 4 and 5 is only directly relevant to linear systems. However, it can be shown that if a linear approximation of the system is stable (unstable), then the true system is also stable (unstable) in the neighborhood of the equilibrium. We present a nonlinear example below.

A step-by-step approach to analyzing systems of differential equations

Here are steps that I use to analyze the dynamics of a system of two differential equations.

There are a variety of approaches to drawing phase diagrams. I find these steps to be quite intuitive and they help me avoid careless mistakes.

1. Find a reduced form for the expressions $\dot{x}_1$ and $\dot{x}_2$ in terms of only x_1, x_2 and exogenous parameters. All other variables must be eliminated from the equations or assumed to be constant.
2. Solve for the inequalities $\dot{x}_1 \geq 0$ and $\dot{x}_2 \geq 0$. This should leave you with two inequalities in terms of x_1 and x_2 that, if satisfied, mean that $\dot{x}_1 \geq 0$ and $\dot{x}_2 \geq 0$.
3. Find the equilibria: the values of x_1 and x_2 such that $\dot{x}_1 = \dot{x}_2 = 0$.
4. Graph the isoclines, i.e., the functions $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ in the (x_1, x_2) plane.
5. Using the inequalities found in 2, determine the trajectories for x_1 and x_2 on either side of the isoclines. That is, on which side of the isoclines is each variable is increasing ($\dot{x}_1 > 0$ and $\dot{x}_2 > 0$) and where are they decreasing ($\dot{x}_1 < 0$ and $\dot{x}_2 < 0$).
   Hint: it is easiest if you carry out steps 4 and 5 separately for each isocline first before putting the two together.
6. Take a linear approximation of the system's dynamics in the neighborhood of each equilibrium and express it as a matrix of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.
7. Check to see if the easy conditions from L&VL (ii & iii above) are satisfied. Then, if necessary, find the Eigen values of this linear system of equations and, following Léonard and Van Long p. 98[6], evaluate the system's stability.

Example

Consider the following example from Léonard and Van Long (p. 102):
x_1 is capital stock and x_2 is the stock of pollution. Capital growth is assumed to be a constant fraction, s, of output, x_1^α with $\alpha<1$, and depreciates at the rate δ, so that the rate of change in capital can be written
$\dot{x}_1 = sx_1^\alpha - \delta x_1$.
The stock of pollution, x_2, grows as a function of capital x_1^β ($\beta>1$) but decays at the rate $\gamma<1$, $\dot{x}_2 = x_1^\beta - \gamma x_2$.

Step 1: $\dot{x}_1 = sx_1^\alpha - \delta x_1$ and $\dot{x}_2 = x_1^\beta - \gamma x_2$

Step 2: Solve for $\dot{x}_1 \geq 0$ and $\dot{x}_2 \geq 0$ and identify associated spaces in the phase diagram.

$\dot{x}_1 \geq 0$		$\dot{x}_2 \geq 0$
$\dot{x}_1 \geq 0 \Rightarrow sx_1^\alpha - \delta x_1 \geq 0$	1	$\dot{x}_2 \geq 0 \Rightarrow x_1^\beta - \gamma x_2 \geq 0$
$sx_1^\alpha \geq \delta x_1$	2	$x_1^\beta \geq \gamma x_2$
$x_1^{\alpha-1} \geq \delta/s$	3	$x_2 \leq x_1^\beta/\gamma$
$x_1 \leq (\delta/s)^{1/(\alpha-1)}$	4	

Note: since $x_1>0$ by assumption, the inequality does not flip when dividing by x_1 at step 3, while since $\alpha-1<0$, the inequality flips from 3 to 4.

[6] Available in the class Google Drive folder.

<u>Step 3</u>: Identify the equilibrium, $\dot{x}_1 = 0 \Leftrightarrow s x_1^{\alpha} - \delta x_1 = 0 \Rightarrow x_1^{\alpha-1} = \frac{\delta}{s} \Rightarrow x_1 = \left(\frac{\delta}{s}\right)^{1/\alpha-1}$

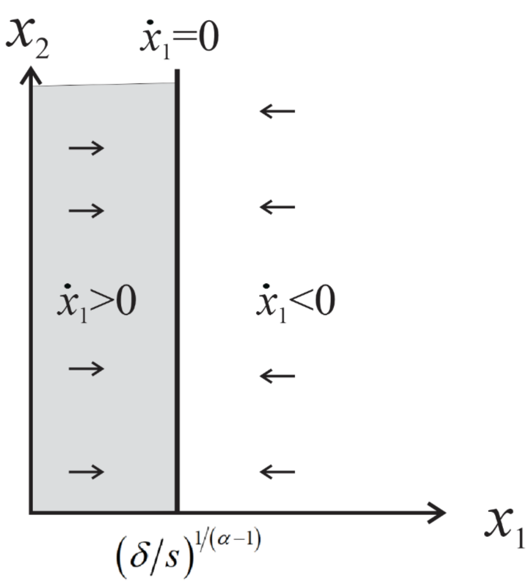
and

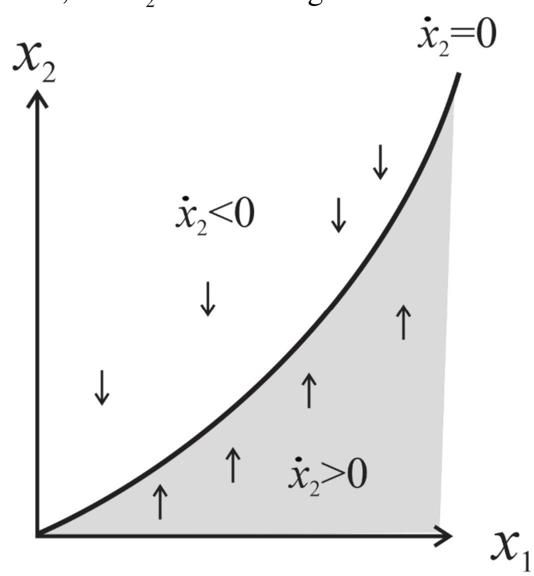
$\dot{x}_2 = 0$ if $x_2 = x_1^{\beta}/\gamma$.

Substituting in the value for x_1 yields $x_2 = \frac{1}{\gamma}(\delta/s)^{\beta/(\alpha-1)}$ so that the equilibrium lies at the point $(\hat{x}_1, \hat{x}_2) = \left(\left(\frac{\delta}{s}\right)^{1/\alpha-1}, \frac{1}{\gamma}(\delta/s)^{\beta/(\alpha-1)}\right)$.

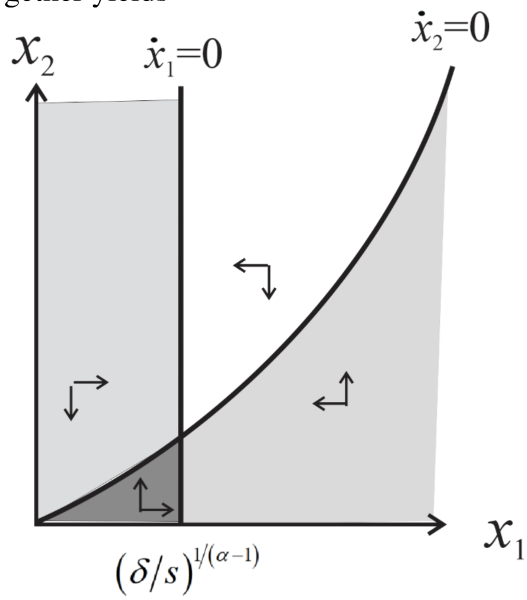
<u>Step 4</u>: Graph the isoclines, $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ $x_1 = (\delta/s)^{1/(\alpha-1)}$ and $x_2 = x_1^{\beta}/\gamma$. (See below)

<u>Step 5</u>: Identify the regions where x_1 and x_2 are increasing and decreasing using the results from the first step: $\dot{x}_1 \geq 0 \Rightarrow x_1 \leq (\delta/s)^{1/(\alpha-1)}$ and $\dot{x}_2 \geq 0 \Rightarrow x_2 \leq x_1^{\beta}/\gamma$. This means that x_1 is increasing to the left of its isocline, and x_2 is increasing below its isocline.





Putting the two together yields



Step 6: Find a linear approximation of the dynamics of the system. To find a first-order Taylor-series approximation of $\dot{x}_1$, recall that if $\dot{x}_1 = f(x_1, x_2)$, then in the neighborhood of the equilibrium, $(\hat{x}_1, \hat{x}_2)$, $\dot{x}_1 \approx f(\hat{x}_1, \hat{x}_2) + \frac{\partial f(\hat{x}_1, \hat{x}_2)}{\partial x_1}(x_1 - \hat{x}_1) + \frac{\partial f(\hat{x}_1, \hat{x}_2)}{\partial x_2}(x_2 - \hat{x}_2)$.

Hence, in this case, when $\dot{x}_1 = sx_1^{\alpha} - \delta x_1$ and $\dot{x}_2 = x_1^{\beta} - \gamma x_2$,

$\dot{x}_1 \approx s\hat{x}_1^{\alpha} - \delta\hat{x}_1 + (\alpha s\hat{x}_1^{\alpha-1} - \delta)(x_1 - \hat{x}_1)$

$\dot{x}_2 \approx \hat{x}_1^{\beta} - \gamma\hat{x}_2 + \beta\hat{x}_1^{\beta-1}(x_1 - \hat{x}_1) - \gamma(x_2 - \hat{x}_2)$.

We know that in most cases in the neighborhood of the equilibrium, the dynamics will be the same as that of the homogeneous system of equations

$$\begin{aligned}\dot{x}_1 &= (\alpha s\hat{x}_1^{\alpha-1} - \delta)(x_1 - \hat{x}_1)\\ \dot{x}_2 &= \beta\hat{x}_1^{\beta-1}(x_1 - \hat{x}_1) - \gamma(x_2 - \hat{x}_2)\end{aligned} \quad \text{or} \quad \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}(\alpha s\hat{x}_1^{\alpha-1} - \delta) & 0\\ \beta\hat{x}_1^{\beta-1} & -\gamma\end{bmatrix}\begin{bmatrix}x_1 - \hat{x}_1\\ x_2 - \hat{x}_2\end{bmatrix}.$$

Step 7: Solving for the Eigen values of the matrix $\begin{bmatrix}(\alpha s\hat{x}_1^{\alpha-1} - \delta) & 0\\ \beta\hat{x}_1^{\beta-1} & -\gamma\end{bmatrix}$, yields $\lambda_2 = -\gamma$, and $\lambda_1 = \alpha s\hat{x}_1^{(\alpha-1)} - \delta$. Plugging in the equilibrium value of $\hat{x}_1 = (\delta/s)^{1/(\alpha-1)}$, this simplifies to $\lambda_1 = \delta(\alpha - 1)$. Since λ_1 and λ_2 are both negative, this implies that we fall in case b, (with opposite arrows) so that the equilibrium is globally stable in the neighborhood.

Thought question: What would happen if production were adversely affected by pollution, i.e., if output took the form $x_1^{\alpha}/\tau x_2$?

Separatrices

As noted above, in some models there exists an important line called a separatrix. These are important economically for they can help us understand how state variables will change over time as they approach an equilibrium or otherwise change over time. Karp (lecture notes) defines a separatrix as “a line in the phase space that trajectories never cross.” The reason this happens is that the slope of a separatrix is the same as the slope of the trajectory. That is, along the separatrix, in the x_1, x_2 plane the separatrix is the set of points along which $\frac{\partial x_1}{\partial x_2} = \frac{\dot{x}_1}{\dot{x}_2}$.

How do we find the separatrix? Recalling that the homogeneous system is set so that the equilibrium is at the origin, this means that we're looking for a function of the form x_2=K x_1 so that $\partial x_2/\partial x_1$=K. We then solve the equation $\frac{\dot{x}_1}{\dot{x}_2} = K$ by simply plugging in the two state equations and solving. In linear systems there are two separatrices.

In nonlinear equations the same basic principle would hold, but the equation would be non-linear (and no doubt more difficult).

X. Reading for next lecture:

Léonard & Van Long pp. 127-151

XI. References

Weisstein, Eric W. “Autonomous.” From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/Autonomous.html. [Originally accessed May 27, 2004].