

Stochastic Dominance Notes AGEC 662

A fundamental concern, when looking at risky situations is choosing among risky alternatives. Stochastic dominance has been developed to identify conditions under which one risky outcome would be preferable to another. The basic approach of stochastic dominance is to resolve risky choices while making the weakest possible assumptions. Generally, stochastic dominance assumes an individual is an expected utility maximizer and then adds further assumptions relative to preference for wealth and risk aversion. We will discuss stochastic dominance in two parts. First, we will review the basic theory then we will cover a number of the extensions that had been done.

1.0 Background to Stochastic Dominance

1.1 Background - Assumptions

There are a number of important assumptions in traditional stochastic dominance.

Assumption #1 - individuals are expected utility maximizers.

Assumption #2 - two alternatives are to be compared and these are mutually exclusive, i.e., one or the other must be chosen not a convex combination of both.

Assumption #3 - the stochastic dominance analysis is developed based on population probability distributions.

1.2 Background - The Expected Utility Basis of Stochastic Dominance

Stochastic dominance assumes expected utility of wealth maximization. Assume x is the level of wealth while $f(x)$ and $g(x)$ gives the probability of each level of wealth for alternatives f and g . We may then write the difference in the expected utility between the prospects as follows.

$$\int_{-\infty}^{\infty} u(x) f(x) dx - \int_{-\infty}^{\infty} u(x) g(x) dx$$

and this equation can be rewritten as:

$$\int_{-\infty}^{\infty} u(x) (f(x) - g(x)) dx \tag{1}$$

If f is preferred to g then the sign of the above equation would be positive. Conversely, if g is preferred to f , the sign of the above equation be negative.

1.3 Background - Integration by Parts

One of the classical calculus techniques for integration is called integration by parts. The basic integration by parts formula is:

$$\int_{-\infty}^{\infty} a db = ab|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} b da$$

where a and b are functions of x .

2.0 Basic Stochastic Dominance

2.1 First Degree Stochastic Dominance

Following the developments in Quirk and Saposnik or Fishburn as reviewed in Anderson, we may apply the integration by parts formula to the last version of the expected utility equation (1). Let us do this by defining an a and b terms which fit the integration by parts structure. Namely, let us choose a to be $u(x)$ and b as the difference between the cumulative density functions as follows:

$$a = u(x)$$

$$b = (F(x) - G(x))$$

where

$$F(x) = \int_{-\infty}^x f(x) dx$$

$$G(x) = \int_{-\infty}^x g(x) dx$$

in turn the differential terms are:

$$da = u'(x) dx$$

$$db = (f(x) - g(x)) dx$$

Notice that under this substitution that adb encompasses the terms in the expected utility equation. Given this substitution the integration of

$$\int_{-\infty}^{\infty} u(x) (f(x) - g(x)) dx$$

equals

$$\left[u(x) (F(x) - G(x)) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u'(x) (F(x) - G(x)) dx$$

We can observe a couple of things about this result. First, let us look at the left hand part. Notice that when the $F(x)$ and $G(x)$ terms are evaluated at x levels of minus infinity they are both zero because we are at the far left hand tail of the probability distribution where the cumulative probabilities equal zero. Thus, the evaluation at minus infinity is zero. Similarly, when x equals plus infinity since these are cumulative probability distributions both will equal one so we have the utility of plus infinity times a term which equals one minus one which is zero. Thus, the left part

of the expression is zero. Now let us look at the right part which is:

$$- \int_{-\infty}^{\infty} u'(x) (F(x) - G(x)) dx$$

Suppose we try to characterize something about the sign of this term. Remember, if the overall sign is positive then f dominates g. We will restrict the sign by adding assumptions. First, suppose that we assume nonsatiation i.e., that more is preferred to less or $u'(x) > 0$ for all x.

Thus, the $u'(x)$ term does not have anything to do with the overall sign of this term as it will always be a positive multiplier. This means this term takes its sign from the $F(x) - G(x)$ term.

That term gives the difference between the two cumulative probability distributions. One can then make a second assumption which is that the difference between $F(x)$ and $G(x)$ is negative or zero for all x. This means that the cumulative probability of distribution of f must always lie on or to the right of the cumulative probability distribution of g (Figure 1). Notice in Figure 1 that for a value of x equal to 7 that there is no meaningful area under the f(x) distribution but there is under the g(x) distribution. Note, for the point x that there is an area under both distributions but that the area underneath the g distribution (i.e., the area between the line and the horizontal axis integrated from the beginning of the probability distribution up to the point x) is greater for the g distribution than it is for the f distribution. Note, when this is true for all x points and therefore we can conclude that f dominates g. What this then does is leads us to the first degree stochastic dominance rule which is as follows:

Given two probability distributions f and g, distribution f dominates distribution g by first degree stochastic dominance when the decision maker has positive marginal utility of wealth for all x

($u'(x) > 0$) and for all x the cumulative probability under the f distribution is less than or equal to the cumulative probability under the g distribution with strict inequality for some x

This requires that for all x the cumulative probability distribution for f is always to the right of the cumulative probability distribution for g or that for every x the cumulative probability of getting that level of wealth or higher is greater under f than under g . Note, the strict inequality requirement means the distribution cannot be the same.

This is not a revolutionary requirement. Some properties are that the mean of f is greater than the mean of g and that for every level of probability you make at least as much money under f as you do under g . This is clearly a very weak requirement, but allows one to characterize the choices between two risky distributions for every utility maximizer that prefers more wealth to less. This is about as weak an assumption as one can make and still resolve some sort of a choice.

2.2 Second Degree Stochastic Dominance

The above stochastic dominance development while theoretically elegant is not terribly useful. What this means is when one is comparing two crop varieties. What one has to observe is that one crop variety always has to consistently perform the other. This may not be the case. The next development in stochastic dominance due to Fishburn; Hanoch and Levy; Hadar and Russell; and Hammond involves making an assumption about risk aversion. We do this by again applying integration by parts and setting the following:

$$a = u'(x)$$

$$db = (F(x) - G(x)) dx$$

so that:

$$da = u''(x) dx$$

$$b = (F_2(x) - G_2(x))$$

where the terms F_2 and G_2 are the second integral of F and G with respect to x , i.e.:

$$\begin{aligned} F_2(x) &= \int_{-\infty}^x \int_{-\infty}^x f(x) dx \\ &= \int_{-\infty}^x F(x) dX \end{aligned}$$

Under these circumstances if we plug in our integration by parts formula we get the equation.

$$- \left[u'(x) (F_2(x) - G_2(x)) \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} u''(x) (F_2(x) - G_2(x)) dx$$

The formula above has two parts. Let us address the right hand part of it first. This contains the second derivative of the utility function multiplied times the difference in the integrals of the cumulative probability distributions with a positive sign in front of it. In order for us to guarantee that f dominates g the sign of this whole term must be positive. Second degree stochastic dominance makes two assumptions that render this term positive. First, assume that the second derivative of the utility function with respect to x is negative everywhere ($u''(x) < 0$). Also, assume that $F_2(x)$ is less than or equal to $G_2(x)$ for all x with strict inequality for some x . Under these circumstances we have a negative times a negative leading to a positive.

We must also sign the left hand part of the above term. First, add the assumption on nonsatiation $u'(x) > 0$. This term then multiplies by $F_2(x) - G_2(x)$ which we know is at plus infinity non-positive since we have already assumed $F_2(x)$ is smaller than $G_2(x)$ while it is zero at x equals minus infinity since there is no area at that stage. This coupled with the leading minus sign

means the whole term will be positive. The second degree stochastic dominance rule can now be stated.

Under the assumptions that an individual has 1) positive marginal utility; ($u'(x) > 0$) 2) diminishing marginal utility of income ($u''(x) < 0$) and 3) that for all x $F_2(x)$ is less than or equal to $G_2(x)$ with strict inequality for some x then we can say that f dominates g by a second degree stochastic dominance.

One aspect of the above assumptions worth mentioning is that when $u''(x)$ is less than zero and $u'(x)$ is greater than zero, this implies the Pratt risk aversion coefficient is positive. Also, the area assumption that the integral under the cumulative probability distribution of f must be smaller than the integral under g allows the cumulative distributions to cross as long as the difference in the areas before they cross is greater than the difference in their areas after they cross.

Figure 2 shows the case where second degree stochastic dominance would exist. Notice the area between g and f before x equals 11 exceeds that after x equals 11. Figure 3 shows a case where stochastic dominance cannot be concluded because of the crossing below x equals 9.

2.3 The Extension to the Third Degree Stochastic Dominance

Whitmore and Hammond made up a third degree stochastic dominance rule by extending this approach once more. They again apply integration by parts. There they find if one assumes that the first derivative is positive, the second derivative negative and the third derivative positive and that the third integral of the probability function of f is always smaller than that of g then f dominates g . The logical intuition is that we have a risk averter with diminishing absolute risk aversion. This test has not been used a great deal in the literature.

2.4 Geometric Interpretation

Suppose we interpret the first and second degree tests geometrically. First degree stochastic dominance requires that the cumulative probability distribution of f always lies to the right or just touching the probability distribution of g . What then happens is the cumulative probability at each income level under g is greater than or equal to the cumulative probability of reaching that income level or less under the f distribution. Conversely one minus that cumulative probability (which is the probability of that income exceeds that level) has to be greater under the f distribution than the g distribution. When the distributions cross first degree dominance is not possible. Thus, at some income levels there is greater probability of exceeding that income level with the g distribution than the f . What second degree does is assume risk aversion and allows the marginal utility of income at the lower levels of wealth exceed to overcome the utility of the additional income increments at the higher levels. What we care about then is the cumulative area between f and g remain positive everywhere or that when f falls below g that it has an advantage and retains that advantage starting from low x values.

2.5 Empirical Implementations

Again, as is the argument in the notes on formation of probability distributions, one does not usually have full continuous probability distributions. Generally, these distribution come about in a discrete fashion. The above presentation is entirely in terms of integrals. Let us now develop the ways of computing the areas in terms of discrete steps. The following procedure develops the probability distributions and the related integrals.

Step 1 - take the wealth or x outcomes for all the probability distributions and array them from high to low as is inherent in tables 1-3.

Step 2 - write the relative frequencies of observations against each of the x levels for each probability distribution. Note, some of these frequencies will usually be zero if for example when an x level is observed under distribution g but not observed under distribution f.

Step 3 - divide the frequencies through by the number of observations under each of the items and if there are 10 observations for f each probability would be the relative frequency times 1/10th.

Step 4 - form the cumulative probability distribution starting at the first x value by taking zero plus the probability of that x for each distribution. For the second and all later x values take the cumulative probability for the prior x plus the relative frequency and accumulate this and at the end both of the cumulative probability distributions should be a one. The algebraic formulae for the area is:

$$F_0 = G_0 = 0$$

$$F_i = F_{i-1} + f_i$$

$$G_i = G_{i-1} + g_i$$

Where the F_i and G_i are the cumulative probability at step i and g and the f_i are the event probabilities.

Step 5 - form the second integral of the probability using the formulae;

$$F_{2,1} = 0$$

$$G_{2,1} = 0$$

$$F_{2,i} = F_{2,i-1} + F_i * (x_i - x_{i-1}) \quad i > 1$$

$$G_{2,i} = G_{2,i-1} + G_i * (x_i - x_{i-1}) \quad i > 1$$

Where F_{2i} and G_{2i} are second integrals at Step i.

An example of this is given in Table 1. Suppose that for distribution f we have one observation at one and three at two, four at four, four at five, two at six, three at seven, two at nine, and one at ten for a total of 20. For g we have two at one, five at two, one at three, five at four, and seven at seven. We then form the distributions as in the Table. Notice for the distribution f we have zero probability of observations at three and eight, whereas for distribution g we have zero probabilities at six through ten. We then form the cumulative probability distribution function as in the cdf columns and put the integral of the cumulatives as in the last two columns in the Table. In this comparison f dominates g by first degree stochastic dominance since every single observation in the cdf column for F is less than or equal to that for G with some strict equalities.

Example 2 presents a case where second degree stochastic dominance holds. First degree fails since for the case of $x = 5$ the cdf for f is greater than the cdf for g . But when we integrate the cumulatives then F_2 is always less than or equal that for G_2 with several strict inequalities.

Example 3 shows a case where dominance does not hold. Note here that the integrated cumulative probability distribution for f is both larger and smaller than that for g . If one looks at this case carefully one can also see that one of the problems with stochastic dominance and that is that the whole reason for the failure of the dominance tests is the low level crossing at $x = 1.95$.

2.6 Moment Based Stochastic Dominance Analysis

One way that stochastic dominance analysis can be done is under distributional assumption. There are a number of derivations of second degree (SSD) results in such cases as reviewed in Pope and Ziemer; Ali; Bawa and Bury. Namely, if one assumes normality then the SSD rule is

$$\mu_f \geq \mu_g$$

$$\sigma_f \leq \sigma_g$$

with at least one strict inequality where μ_f , μ_g , σ_f and σ_g are the mean and variance parameters of the f and g data that is assumed to be normally distributed. Similarly, under log normal distributions we get the rule

$$\mu_f + \frac{\sigma_f^2}{2} \geq \mu_g + \frac{\sigma_g^2}{2}$$

$$\sigma_f \leq \sigma_g$$

and under Gamma distributions we get the rule

$$\frac{\beta_1}{\beta_2} \geq \max(1, \alpha_2/\alpha_1)$$

Each of these rules is discussed in Pope and Ziemer

3.0 Problems With Stochastic Dominance

While stochastic dominance as presented above seems to have nice properties; it has problems inherent in its assumptions and it is not a very discriminating instrument. Let us shed some light on the difficulties and on approaches and procedures that have been advanced to get around them.

3.1 Non-Discrimination - Low Crossings

The first problem is the lack of ability to discriminate among cases with low crossings. Stochastic dominance requires the dominant distribution to always have a greater minimum than the dominated distribution. If the distribution shows a vast improvement under all the observations but the lowest one as in Figure 3 or Table 3, then stochastic dominance will not hold in any form. The real question is how risk adverse will individuals be? Stochastic dominance

assumes that the individuals fall in the class of all risk averters which includes infinitely risk adverse individuals. It assumes someone can possess a risk aversion parameter that is so large that the utility of the small difference at the lowest observation is extraordinary important. The extension to get around this is involves placing bounds on the risk aversion parameter and saying it has to fall in particular numerical ranges.

3.2 Portfolio Effects

A second assumption of stochastic dominance is the assumption that the alternatives are mutually exclusive. When one does stochastic dominance one ignores the possibility that the alternatives could be diversified. This is perfectly reasonable when one is talking about dealing with two mutually exclusive alternatives. On the other hand, if one is looking at acreages of crops to grow an obvious possibility is to not have a monoculture area but rather have a diversified area where one can grow some combination of both. One can use stochastic dominance to look at such questions but one has to form a larger set of mutually exclusive alternatives. For example, 100% corn , 95% corn - 5% cotton , 90% corn - 10% cotton, etc.

3.3 Sample Size

A third problem with stochastic dominance is sampling distributions. Namely, when one goes out and finds data, one does not find population data and one usually finds a sample. For example, data from 5 years of crop yield experiments whereas the true crop could be exposed to a million different years of weather. Stochastic dominance is subject to sampling error and one could for example draw a particular good or bad year or even have some contamination in sample collection.

4.0 Problem Resolution

Each of these problems has had some degree of attention toward its relaxation. We will cover those now.

4.1 Crossings and Dominance Failures

Low crossings is a problem in stochastic dominance so is the existence of crossings in general which cause second degree stochastic dominance rule failure. There have been solutions proposed which make additional assumptions relative to the risk aversion parameters. Two techniques will be reviewed that fall into this class.

4.1.1 Generalized Stochastic Dominance

An extension of stochastic dominance that has been utilized is generalized stochastic dominance (GSD). Here one again starts from the expected utility function:

$$- \int_{-\infty}^{\infty} u'(x) (F(x) - G(x)) dx$$

Meyer investigated the magnitude of this expression under the conditions that the Pratt risk aversion coefficient fall into an interval:

$$r_1(x) \leq \left[\frac{u''(x)}{u'(x)} \right] \leq r_2(x)$$

In this framework what we do is look at the utility difference between f and g but we hold the risk aversion parameter in a particular interval. Meyer poses an optimal control format for this examination

$$\begin{aligned}
\text{Max} \quad & \int_{-\infty}^{\infty} u'(x) (F(x) - G(x)) dx \\
& (u'(x))' = \left(\frac{u''(x)}{u'(x)} \right) u'(x) \\
r_1(x) \leq & - \left[\frac{u''(x)}{u'(x)} \right] \leq r_2(x) \tag{2}
\end{aligned}$$

In this problem Meyer chooses $u(x)$ so as to maximize the utility difference while requiring the risk aversion parameter to be in a particular interval.

When this problem is solved if the solution has a negative objective function value, then under any utility function choice within the $r(x)$ interval, the expected utility criteria will always be positive and therefore f must dominate g . What this says is that when the decision makers utility function has $r(x)$ is in the interval between $r_1(x)$ and $r_2(x)$ that this dominance holds.

Meyer recognized that this is a simple optimal control problem since it is linear in the control variables. The problem has what it is called a Bang-Bang solution. Namely, the solution for $r(x)$ is at either $r_1(x)$ or $r_2(x)$ depending on the criteria. The criteria developed is as follows:

$$r(x^*) = \begin{cases} r_1(x^*) & \text{if } \int_{x^*}^{\infty} u'(x) (F(x) - G(x)) dx > 0 \\ r_2(x^*) & \text{if } \int_{x^*}^{\infty} u'(x) (F(x) - G(x)) dx \leq 0 \end{cases}$$

Which leads to an recursive calculation of the optimal objective function.

$$Q_n = \int_{x_n}^{x_{n+1}} u'(x) (F(x) - G(x)) dx + Q_{n+1}$$

The generalized stochastic dominance rule can now be developed. Namely, f dominates g

whenever the solution of the maxim and in (2) is positive as calculated by the recursive relationship explained above. This is a numerical test based on the data and means that when one goes through numerical evaluation equation for given f and g probability distributions with upper and lower bounds on the risk aversion parameter, that stochastic dominance holds whenever the numerical value of the objective functions comes out positive. Meyer originally wrote a computer program to do this and McCarl has a related program called MEYEROOT on the class web page.

This has been a fairly heavily used technique in risk analysis and virtually everyone that has used it has used constants for $r_1(x)$ and $r_2(x)$ saying that the risk aversion parameter lies somewhere between two absolute numbers. Note this does not imply that the risk aversion parameter is constant but rather that it could be increasing, decreasing or of any other form as long as it remains in between the two bounds. The biggest problem in using that technique has always been to find the r_1, r_2 values. There is a computer program available on McCarl's web page which does do the Meyer calculations. It can use fixed r_1, r_2 or will search for the largest interval given a value for r_1 or r_2 . Namely, when given r_1 it searches for the largest r_2 that permits dominance or when given r_2 finds the smallest r_1 that still permits dominance. However, there are difficulties in how big r_1 and r_2 should be. There is an alternative approach which can be used as discussed next..

Finally, note GSD is a generalization of the other stochastic dominance forms when $r = 0$ and $r_2 = \sigma$ we get σ test equivalent to second degree while $r_1 = -\infty$ and $r_2 = \infty$ is the same as first degree.

4.2 Finding the Discriminating Risk Aversion Parameter - Low Crossings

Yet another approach has been used to deal with crossings. Hammond showed that given two alternatives which cross once that under constant absolute risk aversion there is a break-even risk aversion coefficient (BRAC) that differentiates between those two alternatives. Further, anyone with a risk aversion coefficient (RAC) larger than that particular BRAC will prefer one alternative while any one with a RAC smaller than the BRAC would prefer the other alternative. Hammond's approach has been implemented in two different ways. First and fundamentally, Hammond noted the expected utility problem given an RAC (r)

$$\int_{-\infty}^{\infty} -e^{-rx} f(x) dx$$

is a form of the mathematical statistics moment generating function, i.e., see Hogg and Craig.

Moment generating functions have been derived and are tabled for alternative distributional forms.

Perhaps this should be illustrated with an example. Suppose we assume normality and use the moment generating function for the normal distribution. In this case, the moment generating function given the risk aversion parameter r for distribution f is as follows:

$$m(r) = e^{-(r u_f - \frac{\sigma_f^2 r^2}{2})}$$

If we go to solve this for the break-even risk aversion parameter, first thing we would do is set the expected utilities equal:

$$e^{-(r u_f - \frac{\sigma_f^2 r^2}{2})} = e^{-(r u_g - \frac{\sigma_g^2 r^2}{2})} .$$

In turn if we take the logs of both sides

$$r u_f - \frac{\sigma_f^2 r^2}{2} = r u_g - \frac{\sigma_g^2 r^2}{2}$$

this can be manipulated to

$$r^2 \left(\frac{-\sigma_f^2}{2} + \frac{\sigma_g^2}{2} \right) + (u_f - u_g) r = 0$$

which yields two roots

$$r = 0$$

$$r = \frac{2(u_f - u_g)}{\sigma_f^2 - \sigma_g^2}$$

Notice then for any two normally distributed prospects we can find a break-even risk aversion parameter using this formula just using data on the means and variances.

Therefore, what one can do is take the moment generating function for the f and g distributions then solve for the BRAC which leads to the expected utility being equal. Subsequently, one would investigate the value of the utility difference function above and below that BRAC to come up with a conclusion about which distribution is preferred above and below it. The important difference in this technique relative to Meyers generalized stochastic dominance is rather than having to specify a risk aversion parameter bound one, can solve for the BRAC then proceed to investigate whether it is reasonable for individuals to have risk aversion coefficients which are larger or smaller than that particular value. However, this introduces the problem of knowing the functional form of the assumed probability distribution.

McCarl wrote a program to implement Hammond's approach with an empirical discrete

distribution of unknown form. This program is called RISKROOT and is available on the web page. RISKROOT takes data for two alternatives and searches for the break-even risk aversion parameters between those two alternatives.

This is done by solving the following equation for all applicable values of r .

$$\sum_i - e^{-rx_i} (f(x_i) - g(x_i)) = 0$$

There are several characteristics that are recognized in RISKROOT.

First, Hammond shows the number of roots that can be found is determined by the number of distribution crossings. If there are no distribution crossings then either first degree stochastic dominance must exist and no BRAC can be found. If they cross, then one or more BRAC's may be found.

Second, the number of roots depend upon the number of crossings and if there are 5 crossings it is conceivable there will be 5 BRAC's. Intuitively, a case with multiple BRAC's occurs with distributions with a lower minimum and higher maximum than another but where the other distribution has a higher mean. Note, at extremely high risk aversion, the distribution with a higher minimum will be preferred, while at extremely high risk taking the distribution with the higher maximum will be preferred. However, at moderate levels of risk aversion (somewhere around zero) the distribution with the higher mean will be preferred. One would start preferring the distribution with the higher minimum then switch to the distribution with the higher mean then switch back to the distribution with the higher maximum. Thus, there would be two crossings and one would expect to find two roots.

Third, the maximum size of the BRAC examined by RISKROOT is dependent upon a

formula derived from McCarl and Bessler. It is possible that due to very low or high crossings, a risk aversion parameter cannot be found to be the low or high enough in order to differentiate among the prospects.

Fourth, the BRAC arises from the solution of

$$\sum_i e^{-rx_i} (f(x_i) - g(x_i)) = 0$$

Note that when the risk aversion parameter equals zero the above function becomes zero (since the formula reduces to the sum of the f minus g probabilities equaling one minus one) while when r equals infinity the above function is zero. Thus, (since e^{-rx} goes to zero) there is always a root for risk aversion parameter equal zero and positive infinity.

Fifth, when one uses RISKROOT to find BRAC's one finds results which are limiting results on the Meyer GSD. One cannot span a BRAC within Meyer's framework. Namely, if one found a BRAC of .1 where distribution f is preferred to g above it whereas below it g is preferred to f then if one spans .1 unanimous dominance cannot be found.

4.1.3 Technique Choice

Both the Meyer and GSD technique and the McCarl RISKROOT technique can be used to resolve stochastic dominance choices. We recommend that the McCarl RISKROOT technique be used because it identifies the BRAC points at which preference switches. Let us briefly present an argument from McCarl (1990). At any point for any RAC value one can always find a Meyer interval which will give the same results as the BRAC preferences. Namely, if a BRAC is found at 0.1 above which f is preferred to g and below which the converse is true. Now suppose one wants to investigate what happens at 0.09 is preferred to f then one can find an interval

surrounding 0.09 (it may need to be a small one) where according to the Meyer GSD g is preferred to f . There is no way with GSD τ below 0.1 and 0.2 anywhere above .1 that one can ever find GSD results where f is preferred to g . Thus, the BRACs define the places where the preference shifts. The RISKROOT BRAC gives much stronger results than GSD telling exactly where the preference shifts rather than having to make one hunt for appropriate levels of risk aversion bounds to put into the GSD program.

5.0 Sampling

Pope and Ziemer investigated sampling error. Not a lot can be said beyond the following

1) when distribution means and variances get close together that the probability of improper dominance conclusions can become quite high.

2) using the moment based stochastic dominance rules is inferior to using the empirical distribution based stochastic dominance rules.

3) the smaller the sample size the more likely one is to have errors.

6.0 Portfolios

A problem in stochastic dominance involves potential presence of a portfolios. Namely, one may be looking at two stochastic prospects which are not mutually exclusive but which may be correlated, i.e., if one was looking at two crops one might find that wheat and corn perform differently in different weather conditions because they utilize different growing seasons. Here we investigate the question of what happens with the correlation. The fundamental basis for these notes is in the paper by McCarl, et al. where the portfolio problem is investigated. In that paper several results occur which will not be reviewed here where previous authors have shown conditions under which diversification between two alternatives is optimal. The question that we

deal with involves the conditions under which when one finds that f dominates g that the prospect will also dominate all combinations of the f and g. The procedure for investigation that is used in that paper is based on two moment based stochastic dominance rules.

This portfolio based investigation requires us to take on some additional assumptions so that we may generate analytical results. We will rely on the moment based normality GSD rule which states that normally distributed prospect f dominates normally distributed prospect g whenever the following two conditions are discovered.

$$\begin{aligned} u_f &\geq u_g \\ \sigma_f^2 &\leq \sigma_g^2 \end{aligned}$$

Now we wish to see when prospect f dominates prospect g via the above stochastic dominance rules that we will also find that prospect f will dominate prospect h which is a convex combination of f and g. A convex combination is written according to the following formula:

$$h = \lambda f + (1 - \lambda)g$$

where λ varies between 0 and 1. We also know from mathematical statistics that when we form prospect h that its mean and variance are given by the

$$\begin{aligned} u_h &= \lambda u_f + (1 - \lambda) u_g \\ \sigma_h^2 &= \lambda^2 \sigma_f^2 + (1 - \lambda)^2 \sigma_g^2 + 2\lambda(1 - \lambda)\sigma_f\sigma_g \end{aligned}$$

we also know since f dominates g that the following two equations are satisfied.

$$\begin{aligned} u_f &\geq u_g \\ \sigma_f^2 &\leq \sigma_g^2 \end{aligned}$$

Now what we need to do is investigate the more general dominance conditions between f and h and try and find conditions under which those conditions will hold given some arbitrary f and g. The first rule that we will investigate is the relationship between the means. Notice that the definition of the mean as expressed above allows us to write the following:

$$u_h = \lambda u_f + (1 - \lambda) u_g \leq \lambda u_f + (1 - \lambda)u_f = u_f$$

or

$$u_h \leq u_f$$

This arises since u_g is less than or equal to u_f . Thus, uniformly u_h is always less than or equal to u_f so the first of the two dominance rules is always satisfied.

Examining the second dominance rule is more complicated. Here we need to investigate the relationship between the variance of f and the variance of h. We can get the variance of h from

$$\sigma_h^2 = \lambda^2 \sigma_f^2 + (1 - \lambda)^2 \sigma_g^2 + 2\rho(\lambda)(1 - \lambda) \sigma_f \sigma_g$$

Suppose we make a substitution namely since the variance of f is smaller than the variance of g, we can write.

$$K \sigma_f = \sigma_g \quad K \leq 1$$

which renders our equation into the form

$$\sigma_h^2 = \lambda^2 \sigma_f^2 + (1 - \lambda)^2 K^2 \sigma_f^2 + 2\rho(\lambda)(1 - \lambda)K \sigma_f^2$$

Factoring out the σ_f^2 we get the following or now suppose we get $\sigma_f^2 = \sigma_n^2$ or

$$\sigma_f^2 = \sigma_f^2 (\lambda^2 + (\lambda^2 + (1-\lambda)^2 K^2 + 2\rho K \lambda(1-\lambda)))$$

$$\lambda^2 + (1-\lambda)^2 K^2 + 2\rho \lambda (1-\lambda) K = 1$$

If we collect the square terms in this and use the classical quadratic formula, we find the roots are

$$\lambda = 1$$

and

$$\lambda = \frac{(K^2 - 1)}{(K^2 - 2\rho K + 1)}$$

If we wish to preclude convex combinations we wish the equality of the variances to hold somewhere outside the realm of feasible convex combinations. So what we wish to do is that λ be strictly greater than or equal to one $\lambda \geq 1$. This implies

$$K^2 - 1 \geq K^2 - 2\rho K + 1$$

which can be simplified to

$$2\rho K \geq 2$$

and finally to

$$\rho \geq 1/K = \sigma_1/\sigma_2$$

where ρ is the correlation coefficient. Thus, we have the restriction that the correlation coefficient must be greater than or equal to the ratio of the variances. The significance of this result is that we now have a condition under which we are certain that if f dominates g via a second degree stochastic dominance then f will dominate all potential convex combinations of f and g . This equation has several other implications. Namely, if the items are perfectly correlated

then we are always safe because we know that σ_1 is always less than or equal to σ_2 . Thus, if $\rho = 1$ it is always going to be greater than the ratio of the standard errors. Similarly, if ρ is zero or negative then there is no way that one can ever guarantee that all the convex combinations are dominated. McCarl, et al. do rather extensive evaluation on this rule in mind that it works in a very high proportion of the cases for normal and non-normal cases. They also develop a second criteria for dominance. This starts from a rule the differences in mean. This values use of the certain equivalent for the normal distribution stating f dominates g whenever

$$u_f - \frac{r}{2} \sigma_f^2 \geq u_h - \frac{r}{2} \sigma_h^2$$

under this rule following the same approach as talked through above, they find the following condition f will dominate all combinations of f and g whenever

$$\rho \geq \frac{\sigma_f}{\sigma_g} - \frac{(u_f - u_g)}{r \sigma_f \sigma_g}$$

This can be transformed using the rule that r is twice the value σ as explained in McCarl and Bessler as follows:

$$r = \frac{2Z}{\sigma}$$

to become

$$\rho \geq \frac{\sigma_f}{\sigma_g} - \frac{(u_f - u_g)}{2 Z \sigma_g}$$

what this rule shows is that the maximum acceptable correlation coefficient becomes smaller as the means become more disparate.

What these rules can be used for is to examine when one has two potentially diversified alternatives whether can successfully do dominance analysis between the two without considering diversifications. Namely, if the rules are satisfied one is safe if the rule is not satisfied then one needs to potentially consider diversifications. One can also use the formula for λ as expressed above giving a particular ratio of the standard errors and a correlation coefficient to find the largest possible diversification that should be considered. For example, if one plugs in the ratio of $Z = 2$ where σ_g is twice as big as σ_f , with a correlation of .5 then one can use the formula to find that the diversification that should be considered is something between 100% of f and 43% of f and one then can lay a grid out where one might consider 100, 90, 80, 70, 60, 50 and 43% of f and corresponding values of g and then do stochastic dominance over all those alternatives.

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Table 1. First Degree Stochastic Dominance Example

x	Freq f	Freq g	Pdf f (f_i)	Pdf g (g_i)	CDFf (F_i)	CDF g (G_i)	Intcdf f (F_{2i})	Intcdf g (G_{2i})
1	1	2	0.05	0.10	.05	.10	0.05	0.1
2	3	5	0.15	0.25	.20	.35	0.25	0.45
3		1	0.00	0.50	.20	.40	0.45	0.85
4	4	5	0.20	0.25	.40	.65	0.85	1.5
5	4	7	0.20	0.35	.60	1.00	1.45	2.15
6	2		0.10	0.0	.70	1.00	2.15	3.15
7	3		0.15	0.0	.85	1.00	3.00	4.15
8			0.00	0.0	.85	1.00	3.85	5.15
9	2		0.10	0.0	.95	1.00	4.80	6.15
10	1		0.05	0.0	1.00	1.00	5.80	7.15

Mean 5.2 3.5
Std Err 2.42 1.43

Table 2. Second Degree Stochastic Dominance Example

x	Freq f	Freq g	Pdf f (f_i)	Pdf g (g_i)	CDF f (F_i)	CDF g (G_i)	Intcdf f (F_{2i})	Intcdf g (G_{2i})
1	1	2	0.05	0.10	0.05	0.10	0.05	0.10
2	3	5	0.15	0.25	0.20	0.35	0.25	0.45
3		1	0.00	0.50	0.20	0.40	0.45	0.85
4	4	2	0.20	0.10	0.40	0.50	0.85	1.35
5	4		0.20	0.10	0.60	0.50	1.45	1.85
6	2	7	0.10	0.35	0.70	0.85	2.15	2.70
7	3	3	0.15	0.15	0.85	1.00	3.00	3.70
8			0.00	0.00	0.85	1.00	3.85	4.70
9	2		0.10	0.00	0.85	1.00	4.80	5.70
10	1		0.05	0.00	1.00	1.00	5.80	6.70
Mean	5.2	4.3						
Std Err	2.42	2.29						

Table 3. No Stochastic Dominance

x	Freq f	Freq g	Pdf f (f _i)	Pdf g (g _i)	CDF f (F _i)	CDF g (G _i)	Intcdf f (F _{2i})	Intcdf g (G _{2i})
1.95	1	0	5%	0%	5%	0%	0.10	0.00
2	3	7	15%	35%	20%	35%	0.11	0.02
3		1	0%	5%	20%	40%	0.31	0.42
4	4	2	20%	10%	40%	50%	0.71	0.92
5	4	7	20%	35%	60%	85%	1.31	1.42
6	2		10%	0%	70%	85%	2.01	2.27
7	3	3	15%	15%	85%	100%	2.86	3.27
8			0%	0%	85%	100%	3.71	4.27
9	2		10%	0%	95%	100%	4.66	5.27
10	1		5%	0%	100%	100%	5.66	6.27
Mean	5.25	4.05						
Std Err	2.35	1.86						

Table 4. Example

OUTPUT FROM RISKROOT - CONSTANT RISK AVERSION ROOT FINDER

Example 1

DISTRIBUTION 1 NAME IS CASE 1

DISTRIBUTION 2 NAME IS CASE 2

THE DISTRIBUTIONS DO NOT CROSS -- 1 IS DOMINANT

Example 2

THE DISTRIBUTION CDFS CROSS 2 TIMES

1 HAS BEEN FOUND DOMINANT BETWEEN 0 2.2568226094

1 HAS BEEN FOUND DOMINANT BETWEEN 0 -2.2568226094

Example 3

SUMMARY STATISTICS ON THE DATA

DISTRIBUTION	MEAN	STDDEV	MIN	MAX
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CASE 1	5.25		2.35	1.95 10.00
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CASE 2	4.40		2.01	2.00 7.00
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RAC IS LIMITED TO BE BETWEEN +/- .238779E+01 BASED ON MCCARL AND BESSLER

THE DISTRIBUTION CDFS CROSS 3 TIMES

1 HAS BEEN FOUND DOMINANT BETWEEN 0 2.3877865878

TROUBLE -- FOUND 1 DOMINANT AT HIGHEST RAC

-- SHOULD FIND RAC LARGE ENOUGH THAT 2 DOMINATED

1 HAS BEEN FOUND DOMINANT BETWEEN 0 -2.3877865878

Figure 1. First Degree

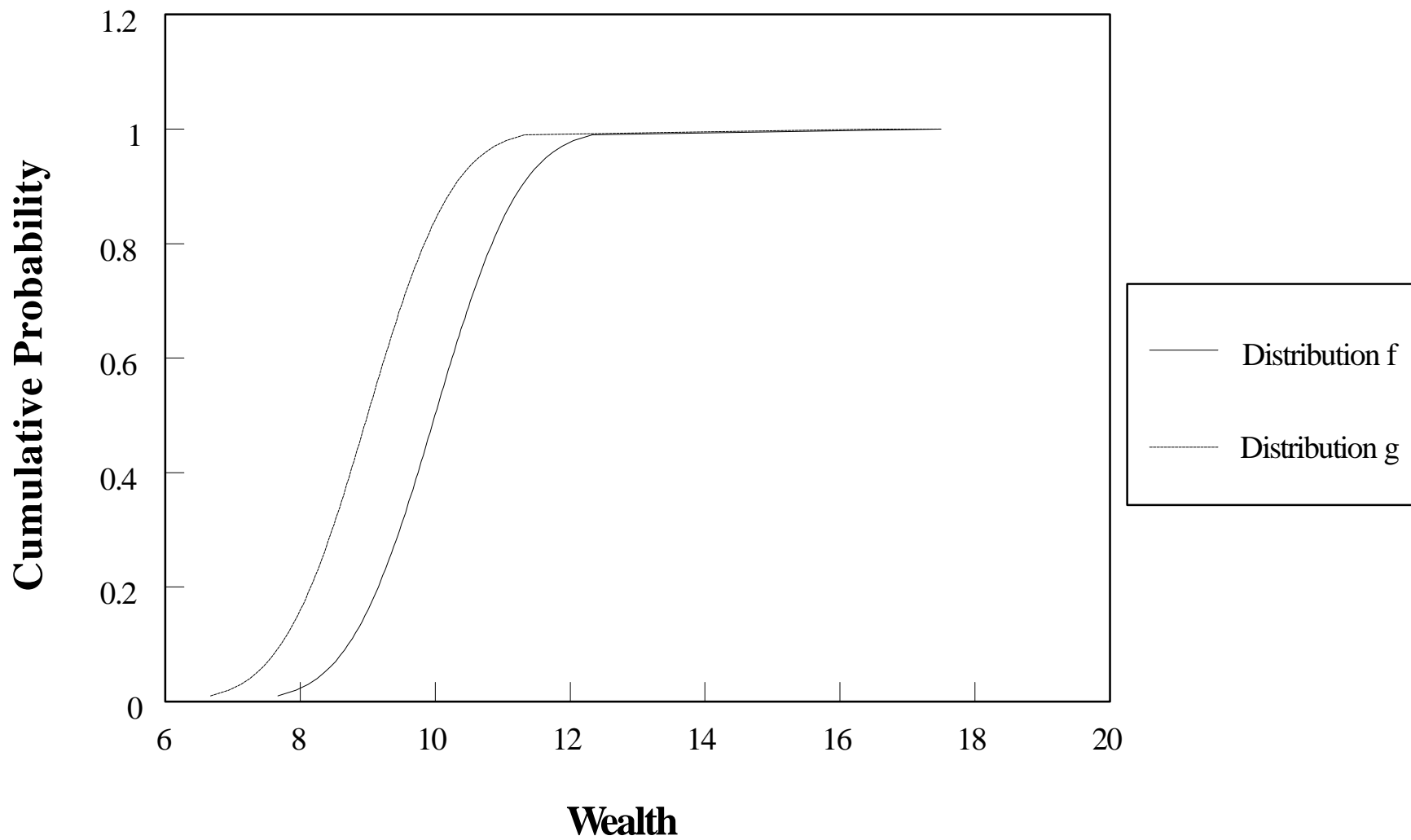


Figure 2. Second Degree

