CHAPTER IX: LINEAR PROGRAMMING MODELING: NONLINEARITIES AND APPROXIMATION

This chapter presents LP formulation techniques for representing nonlinear phenomena. The formulations fall into transformations and approximation classes. Transformations deal with minimization of the sum of absolute values; minimization of the largest absolute value; and maximization of a fraction. Approximations include grid point based formulations of problems with separable and multi-variable functions.

9.1 Transformations

9.1.1 Minimization of the Sum of Absolute Deviations

Suppose one wishes to minimize the sum of absolute deviations between a set of predictions and observations, where the predictions involve endogenously determined variables. Let the deviations be represented by:

$$\in_i \; = \; Y_i - \sum_j X_{ji} \; b_j$$

where i identifies the i^{th} observation, $\in_i$ gives the deviation, Y_i an exogenously observed value, X_{ji} the exogenous data which go into forming the prediction of Y_i, and b_j the endogenous variable levels. The term Y_i minus the sum of X_{ji} b_j gives the difference between the observed level Y_i and its prediction given by (ΓX_{ji} b_j).

A LP constraint set is formed by moving the Γ $X_{ji}b_j$ term to the left side of the equation.

$$\in_i + \sum_j X_{ji} \; b_j \; = \; Y_i$$

The basic problem of minimizing the summed absolute values of all $\in_i$ is:

$$\begin{aligned} \textit{Min} \quad & \sum_i \left|\in_i\right| \\ \textit{s.t.} \quad & \in_i \; + \; \sum_j X_{ji} \; b_j \; = \; Y_i \quad \textit{for all } i \\ & \in_i \; \begin{matrix} < \\ - \\ > \end{matrix} \quad 0 \qquad b_j \; \begin{matrix} < \\ - \\ > \end{matrix} \; 0 \quad \textit{for all } i \textit{ and } j \end{aligned}$$

The variables in this formulation are $\in_i$ and b_j. The $\in_i$ are unrestricted in sign as are the b_j's. The variables in this formulation are $\in_i$ and b_j. The $\in_i$ are unrestricted in sign as are the b_j's.

This problem is not a LP problem because of the nonlinear absolute value function. However, it can be transformed into a LP problem. First, we substitute for $\in_i$, writing it as the difference of two non-negative variables:

$$\in_i = \in_i^+ - \in_i^-$$

$\in_i$ can take on negative values if $\in_i^- > \in_i^+$; conversely, if $\in_i^+ > \in_i^-$, positive values result.

The resultant problem is

$$\textit{Min} \quad \sum_i \left| \in_i^+ - \in_i^- \right|$$

$$\in_i^+ - \in_i^- + \sum_j X_{ji}\, b_j = Y_i \quad \textit{for all } i$$

$$\in_i^+, \in_i^- \geq 0 \qquad b_j \begin{matrix} < \\ - \\ > \end{matrix} 0 \quad \textit{for all } i \textit{ and } j$$

This problem is still nonlinear because of the absolute value term. However the absolute value terms can be simplified whenever either $\in_i^+$ or $\in_i$ equals zero as the consequent absolute value reduces to zero plus the other term. Algebraically, if the product of the deviation variables is zero, i.e.,

$$\in_i^+ * \in_i^- = 0,$$

then the absolute value term can be written as the sum of the two variables

$$\left| \in_i^+ - \in_i^- \right| = \left| \in_i^+ \right| + \left| \in_i^- \right| = \in_i^+ + \in_i^-$$

$$\textit{whenever} \quad \in_i^+ * \in_i^- = 0$$

Imposing the restriction that one or the other variable is zero, the formulation becomes

$$\textit{Min} \quad \sum_i \left(\in_i^+ + \in_i^- \right)$$

$$\textit{s.t.} \quad \in_i^+ - \in_i^- + \sum_j X_{ji}\, b_j = Y_i \quad \textit{for all } i$$

$$\in_i^+ * \in_i^- = 0 \quad \textit{for all } i$$

$$\in_i^+ , \in_i^- \geq 0 \qquad b_j \begin{matrix} < \\ - \\ > \end{matrix} 0 \quad \textit{for all } i \textit{ and } j$$

This is an LP formulation except for the constraint on the product of $\in_i^+$ and $\in_i^-$. However, this constraint can be dropped. Consider a problem with only one observation Y without X and b. Under this case the formulation reduces to

$$\begin{aligned} Min \quad |\epsilon| \quad &= \quad \epsilon^+ + \epsilon^- \\ \epsilon^+ - \epsilon^- \quad &= \quad Y \\ \epsilon^+ \, , \, \epsilon^- \quad &\geq \quad 0 \end{aligned}$$

Rearranging the first constraint we obtain

$$\epsilon^+ = Y + \epsilon^-$$

In turn, tabling alternative values for Y (i.e., consider Y=4, Y=-6), including possible values of ϵ^+ , ϵ^- and the resultant objective function sum yields

Y = 4			Y = -6		
ϵ^+	ϵ^-	$\epsilon^+ + \epsilon^-$	ϵ^+	ϵ^-	$\epsilon^+ + \epsilon^-$
4	0	4*	0	6	6*
16	12	28	14	20	34
Z + 4	Z	2Z + 4	Z	Z + 6	2Z + 6

* These cases are the only ones in which $\epsilon^+ * \epsilon^-$ equals zero.

In the Y=4 case, ϵ^+ has to equal $\epsilon^- +4$. The left most part of the table gives several alternatives for this. The first is $\epsilon^+ =4$ and $\epsilon^- = 0$, leading to a sum ($\epsilon^+ + \epsilon^-$) of 4. The second alternative (16 and 12) gives an objective function sum of 28. In general, for any choice for $\epsilon^- = Z$, the, ϵ^+ value must equal Z+4, and the objective function value becomes 2Z+4. Clearly, when 2Z+4 is minimized and Z is non-negative, the minimum occurs at Z=0, implying $\epsilon^- = 0$. A similar conclusion can be reached for the negative Y case. Thus, minimization will automatically cause $\epsilon_i^+ * \epsilon_i^-$ to equal zero, and the nonlinear constraint is not necessary. Consequently the final formulation becomes

$$\begin{aligned} &\text{Min} \ \sum_i (\epsilon_i^+ + \epsilon_i^-) \\ &\text{s.t.} \quad \epsilon_i^+ + \epsilon_i^- + \sum_j X_{ji} b_j = Y_i \quad \text{for all i} \\ &\qquad \epsilon_i^+, \ \epsilon_i^- \geq 0 \qquad b_j \lesseqgtr 0 \quad \text{for all i and j,} \end{aligned}$$

which is a linear program. This problem solves the original problem. The nonlinear problem has been transformed into an equivalent LP.

9.1.1.1 Example

Suppose a linear equation is to be fit predicting raw orange price as a linear function of the quantity of juice and fresh oranges sold given the following data:

Price of Raw Oranges	Quantity of Oranges Sold	Quantity of Juice Sold
10	8	5
5	9	1
4	10	9
2	13	8
6	15	2
9	17	3

Assume the prediction equation is $Y_i = b_1 X_{i1} + b_1 X_{i2}$, where b_0 is the intercept, b_1 and b_2 are the prediction parameters on the quantity of oranges and juice sold, respectively. Define X_{i1} and X_{i2} as the observations on the quantity of oranges and juice sold, respectively; and Y_i as the observed price. Suppose the desired criteria for equation fit is that the fitted data exhibit minimum sum of the absolute deviations between the raw orange price and its prediction. The formulation would be

$$\begin{aligned} &\textit{Min} && \sum_i |\epsilon_i| \\ &\textit{s.t.} && \epsilon_i = Y_i - b_0 - X_{il}\, b_i - X_{i2}\, b_2 \quad \textit{for all } i \\ & && \epsilon_i \gtrless 0 \qquad b_0, b_1, b_2 \lessgtr 0 \qquad \textit{for all } i \end{aligned}$$

The equivalent LP formulation is

$$\begin{aligned} &\textit{Min} && \sum_i \left(\epsilon_i^+ + \epsilon_i^-\right) \\ &\textit{s.t.} && \epsilon_1^+ - \epsilon_1^- = 10 - b_0 - 8b_1 - 5b_2 \\ & && \epsilon_2^+ - \epsilon_2^- = 5 - b_0 - 9b_1 - 1b_2 \\ & && \epsilon_3^+ - \epsilon_3^- = 4 - b_0 - 10b_1 - 9b_2 \\ & && \epsilon_4^+ - \epsilon_4^- = 2 - b_0 - 13b_1 - 8b_2 \\ & && \epsilon_5^+ - \epsilon_5^- = 6 - b_0 - 15b_1 - 2b_2 \\ & && \epsilon_6^+ - \epsilon_6^- = 9 - b_0 - 17b_1 - 3b_2 \\ & && \epsilon_i^+, \quad \epsilon_i^- \geq 0 \qquad b_0, b_1, b_2 \lessgtr 0 \quad \textit{for all } i \end{aligned}$$

Moving the endogenous variables (i.e., the $\in$'s and b_j's) onto the left-hand side and substituting for the variables which are unrestricted in sign (b_0, b_1, b_2) yields the final formulation given in Table 9.1. The GAMS formulation for this problem is called ABSOLUTE. The objective function minimizes the sum of the deviation variables subject to constraints relating the deviation variables to the difference between the observed and forecast levels for each observation. The coefficients on the intercept are plus ones; the coefficients on the other parameters (b_1, b_2) are the observed levels. The right hand sides are the observed prices to be forecast.

The resultant solution yields an objective function value of 11.277, and the solution is shown in Table 9.2. The predictive equation yielded by this problem reveals that the price of oranges is predicted by the equation 3.426 + (0.191 * the quantity of raw oranges) - (0.149 * the quantity of juice). This equation goes exactly through observations 2, 3, and 5 while nonzero deviations exist for observations 1, 4, and 6. The dual to this problem requires that the shadow prices be between -1 and +1. The dual variables equal these extreme limits when the deviation variables are in the solution. This is shown by the shadow prices on the observations 1, 4, and 6.

9.1.1.2 Comments

The minimization of total absolute deviations formulation has been used in three settings: the solution of regression problems (Charnes, Cooper and Ferguson; Klein; Fisher (1961); Wagner (1959); Wilson); goal programming problems (as covered in the multiple objective chapter), and risk analysis (as in the risk modeling chapter). The regression formulation is commonly used when non-normal errors are expected (see Wilson for discussion).

9.1.2 Minimization of Largest Absolute Deviation

Models can involve minimization of the largest absolute deviation rather than the sum (i.e., the maximum forecast error using the so-called Chebyschev criterion). Such a formulation would be expressed as in the equations

$$\begin{aligned} &\textit{Min} && \left\lfloor \underset{i}{Max} \left| \in_i \right| \right\rfloor \\ &\textit{s.t.} && \in_i \; = \; Y_i \; - \sum_j X_{ji} \; b_j \quad \textit{for all } i \\ & && \in_i, \quad b_j \; \begin{matrix} \leq \\ - \\ > \end{matrix} \; 0 \qquad \textit{for all } i \textit{ and } j \end{aligned}$$

where the variable $\in_i$ is the deviation under the i^{th} observation and b_j is the j^{th} parameter in the forecast equation. The other symbols are as defined in the previous section. The problem formulation is straight forward. Suppose that we define a variable , (with out a subscript) which will equal the largest deviation and introduce two equations for each observation (I):

$$\in \geq \quad Y_i - \sum_j X_{ji}\, b_j$$

$$\in \geq -\left(Y_i - \sum_j X_{ji}\, b_j \right)$$

These equations require $\in$ to be greater than or equal to the deviation and the negative of the deviation for each observation. Thus, $\in$ will be greater than or equal to the absolute deviation from each equation. Taking a simple example without b variables, with observations on Y equaling -3, 2, and 7, then these equations become

Observed Constraints

Y_i	$\in \geq Y_i$	$\in \geq -Y_i$
-3	$\in \geq -3$	$\in \geq 3$
2	$\in \geq 2$	$\in \geq -2$
7	$\in \geq 7$	$\in \geq -7$

Clearly, $\in$ cannot be less than 7 (the largest absolute deviation in the model). Since the objective function minimizes $\in$ subject to these two constraints for each observation, the model collectively minimizes the maximum absolute value. The composite linear program is:

$$\begin{array}{lllll}
\textit{Min} & \in & & & \\
\textit{s.t.} & -\in - & \sum_j X_{ji}\, b_j & \leq & -Y_i \quad \textit{for all } i \\
& -\in + & \sum_j X_{ji}\, b_j & \leq & Y_i \quad \textit{for all } i \\
& \in \geq 0 & b_j \begin{matrix} \leq \\ - \\ > \end{matrix} 0 & & \textit{for all } j
\end{array}$$

9.1.2.1 Example

Utilizing the data from the previous example with the restrictions that the intercept term b_0 is unrestricted in sign but that the parameter on b_1 be non-positive while the parameter b_2 is non-negative. The resultant formulation is

Rows	$\in$	b_0	b_1	b_2		
Objective	1					Minimize
1^+	-1	-1	-8	-5	$\leq$	-10
1^-	-1	1	8	5	$\leq$	10
2^+	-1	-1	-9	-1	$\leq$	-5

2^{--}	-1	1	9	1	≤	5
3^{+}	-1	-1	-10	-9	≤	-4
3^{-}	-1	1	10	9	≤	4
4^{+}	-1	-1	-13	-8	≤	-2
4^{-}	-1	1	13	8	≤	2
5^{+}	-1	-1	-15	-2	≤	-6
5^{-}	-1	1	15	2	≤	6
6^{+}	-1	-1	-17	-3	≤	-9
6^{-}	-1	1	17	3	≤	9
			1		≥	0
				1	≥	0

where all variables are non-negative and the GAMS formulation is called LARGE. This problem solution yields an objective function value of 3.722 with a variable and constraint solution as shown in Table 9.3.

The solution shows the regression line of the price of oranges is equal to 7.167 - 0.111 times the quantity of oranges. The maximum absolute deviation is present at the first, fourth and sixth observations equaling 3.722.

9.1.2.2 Comments

The above formulation solves the Chebyshev criterion problem as discussed in Wagner. This model form results in shadow price sum equaling 1 due to the duality constraint imposed by the form of

$\in$, as observed in the solution. Such a criterion has not been applied widely, but Wilson and Wagner give references.

9.1.3 Optimizing a Fraction

Charnes and Cooper (1962) present a LP formulation involving optimization of a fraction. This formulation allows problems maximizing such things as the average rate of return. The problem is

$$\text{Max} \quad \frac{C_0 + \sum_j C_j X_j}{d_0 + \sum_j d_j X_j}$$

$$\text{s.t.} \quad \sum_j a_{ij} X_j \leq b_i \quad \text{for all i}$$

$$X_j \geq 0 \quad \text{for all j}$$

where the denominator is strictly positive

$$d_0 + \sum_j d_j X_j \succ 0$$

Note there are constants in both the numerator and denominator accounting for exogenous terms which are not a function of the decision variables.

Transformation into a linear program requires several manipulations and substitutions, resulting in an exact transformation of the problem. First, define a variable y_0 which equals one over the denominator

$$y_0 = \left[d_0 + \sum_j d_j X_j \right]^{-1}$$

or equivalently

$$y_0^{-1} = d_0 + \sum_j d_j X_j$$

Multiplying both sides of this relationship by y_0 yields

$$d_0 y_0 + \sum_j d_j X_j y_0 = 1$$

The new variable y_0 is substituted into the above formulation, with the above relationship imposed. The net result is:

$$\begin{array}{lllll} \text{Max} & C_0 y_0 + \sum_j C_j X_j y_0 & & & \\ \text{s.t.} & \sum_j a_{ij} X_j y_0 / y_0 & \le & b_i & \text{for all } i \\ & d_0 y_0 + \sum_j d_j X_j y_0 & = & 1 & \\ & y_0, \quad X_j & \ge & 0. & \end{array}$$

Note that each $a_{ij}X_j$ term has been multiplied by y_0/y_0 which is simply 1. This will be convenient later. Now we introduce a change of variables. Let us define a new variable, y_j equal to the old variable X_j times y_0.

$$y_j = X_j y_0$$

Substituting this into the formulation above yields

$$\begin{array}{lllll} \text{Max} & C_0 y_0 + \sum_j C_j y_j & & & \\ \text{s.t.} & \sum_j a_{ij} y_j / y_0 & \le & b_i & \text{for all } i \\ & d_0 y_0 + \sum_j d_j y_j & = & 1 & \\ & y_0, \quad y_j & \ge & 0. & \end{array}$$

This formulation is not a LP problem; the term y_j/y_0 appears in the first constraint equation.

However,given that y_0 (i.e. the reciprocal of the denominator) is strictly positive we can multiply both sides of the equation through by it without altering the direction of inequality

$$\sum_j a_{ij} y_j \le b_i y_0$$

in turn, rewriting the second equation yields the LP formulation

$$\begin{array}{lllllll} \text{Max} & C_0 y_0 & + & \sum_j C_j y_j & & & \\ \text{s.t.} & -b_0 y_0 & + & \sum_j a_{ij} y_j & \le & 0 & \text{for all i} \\ & d_0 y_0 & + & \sum_j d_j y_j & = & 1 & \\ & y_0, & & y_j & \ge & 0 & \text{for all j} \end{array}$$

which is an exact transformation of the original fractional program. Once this problem has been solved, the levels of the original optimum decision variables are easily discovered by performing the reverse transformation that X_j equals y_j divided by y_0

$X_j = y_j / y_0$.

The LP form includes a new variable with coefficients in the matrix which are the negative of the right hand sides times a new variable ($-b_i\ y_0$). A constraint is also added requiring the constant term in the denominator times the new variable ($d_0\ y_0$) plus the denominator terms involving the transformed variables to equal 1. The transformed model uses the same a_{ij}'s as the original. Its right hand sides are all 0's except the one in the new constraint. The objective function does not have a denominator term and the objective function altered to include the numerator constant times the new variable y_0. Model selection yields the optimal y's ($y_0, y_1, ..., y_n$). Subsequently, then we transform to obtain X.

9.1.3.1 Example

Suppose that it is desirable to solve the following problem.

$$\begin{array}{lllllll} \text{Max} & \frac{}{10} & \frac{}{+} & \frac{1.8X_1}{4X_1} & \frac{+}{+} & \frac{1.7X_2}{4.1X_2} & \\ \text{s.t.} & & & 1.5X_1 & + & X_2 & \le\ 6 \\ & & & 3.0X_1 & + & 4X_2 & \le\ 20 \\ & & & X_1, & & X_2 & \ge\ 0 \end{array}$$

Then the transformed problem is

$$
\begin{array}{lrcrcrcl}
\text{Max} & & & 1.8\,y_1 & + & 1.7\,y_2 & & \\
\text{s.t.} & -6\,y_0 & + & 1.5\,y_1 & + & y_2 & \leq & 0 \\
 & -20\,y_0 & + & 3.0\,y_1 & + & 4\,y_2 & \leq & 0 \\
 & 10\,y_0 & + & 4.0\,y_1 & + & 4.1\,y_2 & = & 1 \\
 & y_0 & & y_1, & & y_2, & \geq & 0
\end{array}
$$

Once a solution to this problem is obtained, the values of the original variables are recovered using the formulas

$X_1 = y_1 / y_0$

$X_2 = y_2 / y_0$

The GAMS model is set up in the file FRACTION and the solution is shown in Table 9.4.

The solution shows that the reciprocal of the denominator equals .031513 and that the decision variables are .042 and .126. Transforming these variables to their original values by dividing them through by the denominator reciprocal yields X_1=1.333 and X_2=4. Plugging back into the original problem, the numerator equals 9.2; the denominator, 31.73, and their fraction 0.29 (the objective function value reported). One may also recover the shadow prices. In this case since the rows are multiplied by one over the denominator, the original shadow prices may be recovered by multiplying through by the denominator as shown in the scaling discussion in Chapter 18 . Thus the effective shadow price for constraint 1 is 10.85, and constraint 2 is 1.33. Constraint 3 has no analogue in the original problem, and thus, the shadow prices are not transformed.

9.1.3.2 Comments

This is an exact transformation as long as the denominator remains strictly positive. The formulation fails if y_0 equals zero in the optimal solution.

Much research has been done on fractional programming. The original development appears in Charnes and Cooper (1962). A historical perspective and literature review can be found in Schaible and Ibaraki.

9.2 Approximations

Approaches to nonlinear problems often utilize approximations. Such approximations may be either one-time or iterative. Discussion of the one-time approximations constitutes the majority of the material below.

9.2.1 Grid Point Approximations

Virtually all one-time approximations use grid points, representing nonlinear phenomena as a discrete series of linearized steps. Such approximations have been used where: a) costs increase with production; b) prices decrease as sales increase; and c) production yields decrease as input usage increases. All these cases involve decreasing returns to scale (increasing returns to scale are covered in the integer programming chapters). Approximations for decreasing

returns cases use a set of discrete grid points assuming that: production cost, output prices, and/or quantities produced are constant between grid points, but change as we move along the grid.

9.2.1.2 Functions with Separable Variables

The most common grid point approximation is separable programming. Separable programming deals with problems in which the functions may be of any nonlinear form, but must be separable into functions of a single variable. For example in the two variable case the functions f(x,y) must be decomposable into h(x) + g(y).

Separable programming is usually considered a nonlinear programming technique (Hadley, 1964); but is commonly used in an LP setting. The most commonly used form of separable programming arose originally with Charnes and Lemke, and was extended by Miller. The formulation yields an LP whenever the objective function terms are concave and the feasible set is convex (Hadley, 1964, p. 124). When these properties do not hold, more general separable programming needs to be used.

Separable programming relies on a set of grid points and constructs an approximation between these points. The approximation is setup so that the approximated value equals the value at the base point plus the slope divided by the difference from the base point. Suppose we wish to approximate the function at point X which falls between approximating points

This can be expressed algebraically by the formula

$$f(X) \cong F(X) = f\left(\hat{X}_k\right) + \frac{f\left(\hat{X}_{k+1}\right) - f\left(\hat{X}_k\right)}{\hat{X}_{k+1} - \hat{X}_k}\left(X - \hat{X}_k\right)$$

In this case, if we write X as a convex combination of $\hat{X}_k$ and $\hat{X}_{k+1}$

$$\begin{array}{lllllll} X & = & \lambda_k & \hat{X}_k & + & \lambda_{k+1} & \hat{X}_{k+1} \\ & & \lambda_k & & + & \lambda_{k+1} & = 1 \\ & & \lambda_k, & & & \lambda_{k+1} & \geq 0 \end{array}$$

where the new variables λ_k and λ_{k+1} are the amount of the k^{th} and $k+1^{st}$ approximation points used.

Substituting this relationship into the above equation for F(X) we get the equation

$$F(X) \cong \lambda_k f\left(\hat{X}_k\right) + \lambda_{k+1} f\left(\hat{X}_{k+1}\right)$$

where the function value is approximated by a convex combination of the function evaluated at the two adjacent grid points. This can be represented by a LP problem. Namely given the separable nonlinear problem

$$\begin{aligned} \text{Max} \quad & \sum_j f_j(X_j) \\ \text{s.t.} \quad & \sum_j g_{ij}(X_j) \le b_i, \quad \text{for all i} \\ & X_j \ge 0 \end{aligned}$$

we may form the approximating problem

$$\begin{aligned} \text{Max} \quad & \sum_j \sum_\mu \lambda_{j\mu} f_j(\hat{X}_{j\mu}) \\ \text{s.t.} \quad & \sum_j \sum_\mu \lambda_{j\mu} g_{ij}(\hat{X}_{j\mu}) \le b_i, \quad \text{for all i} \\ & \sum_\mu \lambda_{j\mu} = 1 \quad \text{for all j} \\ & \lambda_{j\mu} \ge 0 \quad \text{for all j and } \mu \end{aligned}$$

where $\hat{X}_{j\mu}$ is the μ^{th} approximating point for X_j and $X_j = \sum \lambda_{j\mu} \hat{X}_{j\mu}$

This formulation involves a change of variables. The variables $8_{j\mu}$ give the amount of the μ^{th} grid point used in the approximation of the j^{th} variable. The terms $f_j(\hat{X}_{j\mu})$ and $g_{ij}(\hat{X}_{j\mu})$ give the values of the objective function and constraint terms evaluated at the various grid points. The new constraints on the λ variables cause a convex combination of the grid points to be chosen for each variable approximated. The functions must be properly behaved, otherwise the nonzero λ's in the solution will not necessarily be adjacent; and the approximation will not work properly (Hadley, 1964). That is why users of the approximation should be careful to ensure that diminishing returns to production are present whenever this approach is being used.

9.2.1.1.1 Example 1.

Suppose we approximate the problem.

$$\begin{aligned} \text{Max} \quad & (4-.25X)\ X - (1+.25Z)\ Z \\ & X - 3Y \le 0 \\ & 2Y - Z \le 0 \\ & X, \quad Y, \quad Z \ge 0 \end{aligned}$$

To set this problem up, suppose we use values of X equal to 1,2,3,4,5,6 and the same values for Z. The separable programming representation is in Table 12.

Note that λ_2 stands for the amount of the gridpoint X=2 utilized having an objective value equal to the nonlinear function of X evaluated at X=2. The GAMS formulation is called SEPARABL and the resultant solution is shown in Table 9.5. The objective function value is 7.625. The model sets $\lambda_4 = \lambda_5 = 0.5$ amounting to 50% of gridpoint X_4 and 50% of X_5 or X=4.5. The value of Y = 1.5. Simultaneously $\beta_1 = 1$ implying Z = 3. Now, let us examine the adequacy of the

approximation. The objective function approximation for X has 12(.5) + 13.75(.5) = 12.875, while the true f(X) = 12.9375. The Z approximation has zero error in this case. The modeler could either accept this as an adequate approximation or enter new grid points in the neighborhood of this solution.

9.2.1.1.2 Example 2: Separable Terms in the Constraints

The above example deals with the approximation of separable objective function terms which McCarl and Onal found computationally unattractive. On the other hand, separable programming can also approximate constraint nonlinearities, which McCarl and Onal found attractive.

Suppose we wish to approximate the following problem

$$\begin{array}{lrcl} \text{Max} & 3X - 3Y & & \\ \text{s.t.} & X - (20 + 2Y - .2Y^2) & \leq & 0 \\ & X, \quad Y & \geq & 0 \end{array}$$

Selecting a grid for Y of 0, 1, 2, 3, 4 and 5, the separable programming formulation becomes

$$\begin{array}{lrcl} \textit{Max} & 3X - 0\lambda_1 - 3\lambda_2 - 6\lambda_3 - 9\lambda_4 - 12\lambda_5 - 15\lambda_6 & & \\ \textit{s.t.} & X - 20\lambda_1 - 21.8\lambda_2 - 23.2\lambda_3 - 24.2\lambda_4 - 24.8\lambda_5 - 25\lambda_6 & \leq & 0 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 & = & 1 \\ & X, \quad \lambda_1, \quad \lambda_2, \quad \lambda_3, \quad \lambda_4, \quad \lambda_5, \quad \lambda_6 & \geq & 0 \end{array}$$

The resultant GAMS model is in the file CONSEPAR and the solution is given in Table 9.6. We may plug this solution back into the original problem to evaluate the adequacy of the approximation. The values of λ imply that Y equals 3. However, optimization using calculus shows the optimum to be at Y equals 2.5, giving a yield of 23.75 and profits of 63.75. Thus, this demonstrates a 0.235 percent error of approximation. Again, one could go on to add more grid points, or accept the approximation error.

9.2.1.2 Gridpoints and Gridpoint Refinements

The separable formulation uses gridpoints to approximate functions. Readers may wonder how to define such points. Gridpoints are always defined in the context of the applied problem. The gridpoints should provide a reasonable approximation of the function in the domain of the answer, including points both close to the expected answer as well as points depicting functional extremes (Geoffrion (1977) discusses the importance of the extreme points). Even spacing of the gridpoints is not required. Guder and Morris show minimum theoretical error occurs with equal spacing. Thus, one could approximate a curve at the points 10, 2, 1, .95, .50, .10, .02 and .01. The gridpoint also may be redefined given a solution where, for example, one might find a solution of X = 2.50, discover the approximation is inadequate at that point, and then enter more gridpoints in the neighborhood of 2.5. Gridpoint refinement schemes are discussed in Bazaraa and Shetty. Implementation of a gridpoint refinement scheme is discussed in Kochenberger, Woolsey and McCarl.

9.2.1.3 Gridpoint Approximation of Functions of Multiple Variables

Gridpoint approximation may also be applied to functions containing multiple variables. In this case a multi dimensional grid is defined. This approach generally only works when one is approximating functions that depict a concave objective function and a convex constraint set.[1] The method involves techniques similar to separable programming and was developed by Dantzig and Wolfe. This method is discussed in Duloy and Norton; Shapiro (1979b); Bradley, Hax and Magnanti; and Lasdon. One of the possible formulations on this scheme which can be used is

$$\begin{array}{llllll} \text{Max} & CX & - & \sum_j d_j Y_j & & \\ \text{s.t.} & X & - & H(Y_1, Y_2, Y_3, ..., Y_n) = & 0 & \\ & & & Y_j \leq & b_i & \text{for all } j \\ & X, & & Y \geq & 0 & \end{array}$$

where there are multiple inputs and one output (for simplicity). The output X is a function of the levels of the multiple inputs (Y_j). Also the function $H(Y_1...Y_n)$ has to be such that this problem has a convex constraint set.

We will discuss two versions of this formulation. The first version deals with cases where H is homogeneous of degree one and the second where H is homogeneous of degree less than one.

9.2.1.3.1 Homogeneous of Degree 1

The function H being homogeneous of degree 1 implies that

$$H(\alpha Y) = \alpha H(Y)$$

Suppose we choose a set of rays $\hat{Y}_{ju}$ which depict the way each Y_j participates in each ray and define the variable α_u which is the amount of ray $\hat{Y}_{ju}$ which is used. Then we know that

$$H(\alpha_u \hat{Y}_{iu}) = \alpha_u H(\hat{Y}_{ju})$$

i.e., the function α times the ray values equal α times the function evaluated at the base ray point. The generalized programming formulation then becomes

[1] Readers unfamiliar with concavity and convexity should look at the Non-Linear Programming Theory chapter.

$$
\begin{array}{llllll}
Max & CX & & - \sum_j d_j Y_j & & \\
s.t. & X - \sum_u \alpha_u H(\hat{Y}_{ju}) & & & = 0 & \\
 & \sum_u \alpha_u Y_{ju} & - Y_j & = 0 & for\ all\ j & \\
 & & Y_j & \le b_j & for\ all\ j & \\
 & X, \quad \alpha_u, & Y_j & \ge 0 & for\ all\ u\ and\ j &
\end{array}
$$

The approximating model has the rays represented by a variable indicating how much of a particular ray Y_{ju} combination is used. They should be defined with unique ratios of the variables within

$\hat{Y}_{ju}$

(i.e., 1:1, 1:4, 4:1, etc., as below).

Example

This formulation is probably best illustrated by example. Consider the problem

$$
\begin{array}{llllll}
Max & 4X - 20Y_1 - 100Y_2 & & & & \\
s.t. & X - 21Y_1^{0.75}Y_2^{0.25} & & & = 0 & \\
 & Y_1 & & & \le 50 & \\
 & X, \quad Y_1, & & Y_2 & \ge 0 &
\end{array}
$$

Suppose we select a set of combinations for Y_1 and Y_2, given in Table 9.7, showing that when 1 unit

of Y_1 and 4 units of Y_2 are used, X=29.7. Similarly, when 8 units of Y_1 and 8 units of Y_2 are used, X=168. The resultant formulation is

$$
\begin{array}{llllllll}
Max & 4X & & & & - 20Y_1 & - 100Y_2 & \\
s.t. & X - 29.7\alpha_1 & - 168\alpha_2 & - 59.4\alpha_3 & & & & \le 0 \\
 & \alpha_1 + & 8\alpha_2 + & 4\alpha_3 - & Y_1 & & & = 0 \\
 & 4\alpha_1 + & 8\alpha_2 + & \alpha_3 & & - Y_2 & & = 0 \\
 & & & & Y_1 & & & \le 50 \\
 & X, \quad \alpha_1, & \alpha_2, & \alpha_3, & Y_1, & & Y_2 & \ge 0
\end{array}
$$

An isoquant graph of this situation is portrayed in Figure 9.1.

Note that the three lines in the graph stand for the combinations 4 to 1, 1 to 1 and 1 to 4. The connected line in the graph is the isoquant for output equals 168, and the linear segments show how the production process is represented.

The GAMS formulation of the problem is called HOMOGEN and the solution is given in Table 9.8. This solution implies input use in the ratio 4:1. We may wish to put more rays in the neighborhood of 4:1 or we may be willing to accept the approximation error.

9.2.1.3.2 Homogeneous of Degree Less Than One

Now we turn to the case where we do not have homogeneity of degree one. In this case, the function evaluated at α times the vector of inputs Y, is less than α times that functional value evaluated at one unit at Y providing α is less than or equal to one.

$$H(\alpha Y) \; < \; \alpha H(Y)$$

Consider the multiplicative function

$$X = a Y_1^{b_1} Y_2^{b_2} \ldots Y_n^{b_n} = a \pi_j x_j^{b_j}.$$

We may set up a vector representation

$$Y_j = \alpha_u Y_{ju}$$

Under this substitution the function becomes

$$X = a(\pi_j Y_{ju}^{b_j}) \alpha_u^{\sum b_j}$$

But, the sum of the exponents on α_u is less than one

$$\sum_j b_j \prec 1.$$

Thus, as one moves α units along the ray the function only increases by a factor of $\forall$ to the Σ b_j which results in an increase less than α X. This problem exhibits diminishing returns to scale because as α_u is increased, less and less output is produced per unit increase in α_u. This particular problem, however, may be formulated as a linear problem. This approach has been called the "column generation method" as discussed in Shapiro (1979b). Specifically, suppose we choose grid points Y_{ju} and a set of a priori multipliers α_{uL}. The problem then becomes

$$\begin{aligned}
&\textit{Max} \quad CX \; - && \sum_j d_j Y_j \\
&\textit{s.t.} \qquad X \; - \; \sum_u \sum_L H(Y_{ju}\alpha_{uL})\lambda_{uL} && = \; 0 \\
&\qquad\qquad \sum_u \sum_L (Y_{ju}\alpha_{uL})\lambda_{uL} \; - && Y_j \; = \; 0 \quad \textit{for all } j \\
&\qquad\qquad \sum_u \sum_L \lambda_{uL} && = \; 1 \\
&\qquad\qquad && Y_j \; \leq \; b_j \quad \textit{for all } j \\
&\qquad X, \qquad\qquad \lambda_{uL}, && Y_j \; \geq \; 0
\end{aligned}$$

The variables are λ_{uL} where u identifies the input combination and L the length along that input combination. The parameter α_{uL} gives how far along the uth ray we move. The sum of the λ variables are then equal to one. This is a combination of the separable programming and homogeneity of degree one formulations above.

Example

Consider the example problem

$$\begin{array}{lccccccc} \text{Max} & 0.5X & - & 2Y_1 & - & 2Y_2 & & \\ & X & - & 21Y_1^{.5}Y_2^{.25} & & & = & 0 \\ & & & Y_1 & & & \leq & 10 \\ & & & Y_1, & & Y_2 & \geq & 0 \end{array}$$

where the exponents sum to 0.75 so the function is homogeneous of degree less than one. If we then put in three different approximation rays 1 to 1, 1 to 2 and 2 to 1 in these cases, the resultant values of X are

$$X = 21\alpha_1^{.75}, \quad X = 29.7\alpha_2^{.75}, \quad X = 25\alpha_3^{.75}$$

We obtain a function that along the lines X is equal to some constant times $\alpha^{0.75}$. We then develop a table of approximations (Table 9.9). The resultant formulation is in Table 9.10 and in the file NONHOMOG. Note, here we have four combinations for each ratio of inputs, each representing different multiples of α_{uL}. The convexity constraint is needed to insure that the model uses no more than one unit of the first step and rather is forced to go into the latter steps of the production process. The solution of the problem is given in Table 9.11 and shows that the 4th step of the second ray is used resulting in the value for the variables of X=99.3, Y_1= 10, Y_2= 5 with the objective function equal to 19.65.

9.2.1.3.3 Comments

We get many classroom questions as to why we have presented the above generalized approximation formulations. There are two reasons. First, they constitute an approximation that can be used when representing a relationship between multiple inputs and outputs (i.e., see Onal et al.). Such a case occurs in agricultural models when approximating fertilizer response functions containing two or more fertilizer inputs or when intercropping is modeled. Approximations have also involved more complex production functions, where the output is a function of multiple inputs. Second, following Dorfman (1953), this can be used as a conceptual model. Often modelers include a number of activities for the production of a good where the input combinations and outputs arise from experiments or observed behavior. In this case, one is representing the underlying production process without ever estimating it. Such a procedure is utilized in Erhabor and Job.

A second question involves the manner in which grid points are chosen. Again, as in the separable programming case, this is done in accordance with the problem. For example, when one knows common levels of input use, one might construct several combinations of deviations from these numbers in small increments. Thus, when fertilizer and herbicide are used commonly in the ratio 50 lbs. fertilizer to 1 gallon herbicide one might add 5 activities involving: 1) 1 gal. herbicide with 50 lbs. of fertilizer; 2) 1 gal. herbicide with 47.5 lbs. of fertilizer; 3) 1 gal. herbicide with 45 lbs. of fertilizer; 4) 1 gal. herbicide with 52.5 lbs. of fertilizer and 5) 1 gal. herbicide with 55 lbs. of fertilizer. In turn, the user should examine the model solution and see if the solutions chosen use the most extreme ray for an input (e.g., the least amount of herbicide possible per unit of fertilizer). In such a case one should consider entering alternatives expanding the space represented. The representation is only satisfactory when the solution is interior to the cone of approximation points used and not on its boundary (Shapiro and Geoffrion provide theoretical explorations of related topics).

9.2.2 Iterative Approximations

In addition to the step approximation formulations above, there are a number of iterative approximations which can be used. We will not cover these in depth; they are largely numerical techniques. Those interested in alternative approaches should examine Zangwill's convex simplex method, Dembo's geometric programming condensation method, or the methods reviewed in Himmelblau; Reklaitis et al; or Bazaraa and Shetty.

We will explain one technique for illustrative purposes. The iterative approximation presented here was developed by Griffith and Stewart and is based on the concept of a Taylor series expansion. This method solves the problem

$$\begin{aligned} Max \quad & f(X) \\ & g(X) \leq b \\ & L_j \leq X_j \leq G_j \end{aligned}$$

using a first order Taylor series expansion. A first order Taylor series expansion assumes that a functional value can be represented as a first order expansion of the function evaluated at a base point plus the derivative of that base point times the difference of X from the base point. The approximating problem then is given by

$$\begin{aligned} Max \quad & f(X_0) + \frac{d}{dX} f(X_0)\ (X - X_0) \\ s.t. \quad & g_i(X) = g_i(X_0) + \frac{d}{dX} g_i(X_0)\,(X - X_0) \leq b_i \\ & L_j \leq X_j \leq G_j \end{aligned}$$

where given a base point X_0 we approximate the value at any X using a LP formulation to find the difference from X_0 that the solution will move where all of the terms involving X_0 are constants. This is done by substituting in a variable μ_j such that

$\mu_j = X_j - X_{0j}$

to obtain the LP problem

$$Max \quad f(X_o) + \sum_j \frac{d}{dX_j} f(X_o)\mu_j$$

$$\sum_j \frac{d}{dX_j} g(X_o)\mu_j \leq b_i - g_i(X_o)$$

$$-Lim_j^- \leq \mu_j \leq Lim_j^+$$

where the limits are developed relative to an exogenous parameter β_j

$$Lim_j^- = \min\left[\beta_j,\ X_{oj} - L_j\right]$$
$$Lim_j^+ = \min\left[\beta_j,\ G_j - X_{oj}\right]$$

Here the variables are given by

$$X_{0j}^{k+1} = X_{0j}^k + \mu_j^*$$

Then, given any initial choice of variables at the k^{th} iteration, the variable at $k^{th}+1$ iteration is equal to that variable at the k^{th} iteration plus the optimal change variable value

μ_j^* The change variables are artificially constrained to be limited by some quantity β_j. It is desirable that this quantity becomes smaller as iterations proceed.

9.2.3 Other Approximations

We have covered only a few of the approximations which are possible in the area of nonlinear programming. There are also other approximations based on exotic transformations for various sorts of problems; e.g., see Dembo; or McCarl and Tice. Many approximations may be used given special problem structures. Their use depends on the ingenuity of the modeler. What we have attempted to do above is give some of the basic techniques and references.

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Table 9.1. Minimization of Sum of Absolute Deviations Formulation

	ϵ_1^+	ϵ_1^-	ϵ_2^+	ϵ_2^-	ϵ_3^+	ϵ_3^-	ϵ_4^+	ϵ_4^-	ϵ_5^+	ϵ_5^-	ϵ_6^+	ϵ_6^-	b_0	b_1	b_2	
Obj	1	1	1	1	1	1	1	1	1	1	1	1				Min
1	1	-1											1	8	5	=10
2			1	-1									1	9	1	=5
3					1	-1							1	10	9	=4
4							1	-1					1	13	8	=2
5									1	-1			1	15	2	=6
6											1	-1	1	17	3	=9

Table 9.2. Solution of Minimization of Absolute Deviation Sum Example

Objective function = 11.277

Variable	Value	Reduced Cost	Equation	Slack	Shadow Price
$\in_1^+$	5.787	0	Obs 1	0	1
$\in_1^-$	0	2.000	Obs 2	0	-0.660
$\in_2^+$	0	1.66	Obs 3	0	0.191
$\in_2^-$	0	0.340	Obs 4	0	-1
$\in_3^+$	0	0.809	Obs 5	0	-0.532
$\in_3^-$	0	1.191	Obs 6	0	1
$\in_4^+$	0	2.000			
$\in_4^-$	2.723	0			
$\in_5^+$	0	1.532			
$\in_5^-$	0	0.468			
$\in_6^+$	2.766	0			
$\in_6^-$	0	2.000			
b_0	3.426	0			
b_1	0.191	0			
b_2	-0.149	0			

Table 9.3. Solution of Largest Absolute Deviation Example

Variables	Value	Reduced Cost	Equation	Slack	Shadow Price
$\in$	3.722	0	1^+	0	-0.222
b_0	7.167	0	1^-	7.44	0.0
b_1	-0.111	0	2^+	4.89	0.0
b_2	0.000	2.056	2^-	2.56	0.0
			3^+	5.78	0.0
			3^-	1.67	0.0
			4^+	7.44	0.0
			4^-	0	-0.5
			5^+	3.22	0.0
			5^-	4.22	0.0
			6^+	0	-0.278
			6^-	7.44	0.0

Table 9.4. Solution to the Example for Optimizing a Fraction

Objective function = 0.2899

Variable	Value	Reduced Cost	Equation	Slack	Shadow Price
y_0	0.032	0	1	0	0.342
y_1	0.042	0	2	0	0.042
y_2	0.126	0	3	0	0.290

Table 9.5. Solution to the Step Approximation Example

Objective function = 7.625					
Variable	Value	Reduced Cost	Equation	Slack	Shadow Price
λ_1	0	-3.000	1	0	1.750
λ_2	0	-1.500	2	0	2.625
λ_3	0	-0.500	3	0	5.000
λ_4	0.5	0	4	0	2.625
λ_5	0.5	0			
λ_6	0	-0.500			
Y	1.5	0			
β_1	0	-1.250			
β_2	0	-0.375			
β_3	1	0			
β_4	0	-0.125			
β_5	0	-0.750			
β_6	0	-1.875			

Table 9.6. Solution to the Constraint Step Approximation Problem

Objective function = 63.6					
Variable	Value	Reduced Cost	Equation	Slack	Shadow Price
X	23.2	0	1	0	3
λ_1	0	-3.6	2	0	63.6
λ_2	0	-1.2			
λ_3	1	0			
λ_4	0	0			
λ_5	0	-1.2			
λ_6	0	-3.6			

Table 9.7. Set of Y_1, Y_2 Combinations for Homogeneous of Degree 1 Example

X	Y_1	Y_2
29.7	1	4
168	8	8
59.4	4	1

Table 9.8. Solution to Example Problem for Homogeneous of Degree 1

Objective function = 719.8

Variable	Value	Reduced Cost	Equation	Slack	Shadow Price
X	742.5	0	1	0	4
α_1	0	-315.6	2	0	34.4
α_2	0	-403.2	3	0	100
α_3	12.5	0			
Y_1	50	14.4			
Y_2	12.5	0			

Table 9.9. Approximations for the Homogenous of Degree Less Than One Example

X	Y_{11}	Y_{12}	X	Y_{12}	Y_{22}	X	Y_{13}	Y_{23}
21	1	1	29.7	2	1	25.0	1	2
59.4	4	4	49.9	4	2	70.6	4	8
80.5	6	6	67.7	6	3	95.7	6	12
118.1	10	10	99.3	10	5	140.4	10	20

Table 9.10. Formulation of the Homogeneous Degree Less than One Example

Rows	X	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{21}	λ_{22}	λ_{23}	λ_{24}	λ_{31}	λ_{32}	λ_{33}	λ_{34}	Y_1	Y_2	RHS
Obj	0.5													-2	-2	max
x bal	1	-21	-59.4	-80.5	-118.1	-29.7	-49.9	-67.7	-99.3	-25.0	-70.6	-95.7	-140.4			= 0
Y bal		1	4	6	10	2	4	6	10	1	4	6	10	-1		= 0
		1	4	6	10	1	2	3	5	2	8	12	20		-1	= 0
convex		1	1	1	1	1	1	1	1	1	1	1	1			≤ 1
Y lim														1		≤ 10

Table 9.11. Solution to the Homogenous of Degree Less Than One Example

Objective function = 19.651					
Variable	Value	Reduced Cost	Equation	Slack	Shadow Price
X	99.3	0	1	0	0.500
λ_{11}	0	-5.506	2	0	2.850
λ_{12}	0	-0.856	3	0	2.000
λ_{13}	0	0.000	4	0	11.156
λ_{14}	0	-0.606			
λ_{21}	0	-4.006			
λ_{22}	0	-1.581			
λ_{23}	0	-0.404			
λ_{24}	1	0.000			
λ_{31}	0	-5.519			
λ_{32}	0	-3.237			
λ_{33}	0	-4.384			
λ_{34}	0	-9.434			
Y_1	10	0.850			
Y_2	5	0.000			

Figure 9.1 Approximation of Homogeneous of Degree One Example