CHAPTER II: LINEAR PROGRAMMING

The most fundamental optimization problem treated in this book is the linear programming (LP) problem. In the LP problem, decision variables are chosen so that a linear function of the decision variables is optimized and a simultaneous set of linear constraints involving the decision variables is satisfied.

2.1 The Basic LP Problem

An LP problem contains several essential elements. First, there are decision variables (X_j) the level of which denotes the amount undertaken of the respective unknowns of which there are n (j=1, 2 ..., n). Next is the linear objective function where the total objective value (Z) equals $c_1X_1 + c_2X_2 + \cdots + c_nX_n$. Here c_j is the contribution of each unit of X_j to the objective function. The problem is also subject to constraints of which there are m. An algebraic expression for the i^{th} constraint is $a_{i1}X_1 + a_{i2}X_2 + \cdots + a_{in}X_n \leq b_i$ (I=1, 2, ..., m) where b_i denotes the upper limit or right hand side imposed by the constraint and a_{ij} is the use of the items in the i^{th} constraint by one unit of X_j. The c_j, b_i, and a_{ij} are the data (exogenous parameters) of the LP model.

Given these definitions, the LP problem is to choose $X_1, X_2, ..., X_n$ so as to

$$
\begin{array}{lcccccccc}
Max & c_1 X_1 & +c_2 X_2 & +c_3 X_3 & \ldots & +c_n X_n & & \\
s.t. & a_{11} X_1 & +a_{12} X_2 & +a_{13} X_3 & \ldots & +a_{1n} X_n & \leq & b_1 \\
 & a_{21} X_1 & +a_{22} X_2 & +a_{23} X_2 & \ldots & +a_{2n} X_n & \geq & b_2 \\
 & a_{21} X_1 & +a_{32} X_2 & +a_{33} X_3 & \ldots & +a_{3n} X_n & = & b_3 \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
 & a_{m1} X_1 & +a_{m2} X_2 & +a_{m3} X_3 & \ldots & +a_{mn} X_n & \leq & b_m \\
 & X_1 & X_2 & X_3 & & X_n & \geq & 0
\end{array}
$$

This formulation may also be expressed in matrix notation.

$$
\begin{array}{llcc}
\text{Max} & CX & & \\
\text{Subject to} & AX & \leq & b \\
 & X & \geq & 0
\end{array}
$$

Many variants have been posed of the above problem and applications span a wide variety of settings. For example, the basic problem could involve setting up: a) a livestock diet determining how much of each feed stuff to buy so that total diet cost is minimized subject to constraints on minimum and maximum levels of nutrients; b) a production plan where the firm chooses the profit maximizing level of production subject to resource (labor and raw materials) constraints; or c) a minimum cost transportation plan determining the amount of goods to transport across each available route subject to constraints on supply availability and demand.

2.2 Basic LP Example

For further exposition of the LP problem it is convenient to use an example. Consider the decision problem of Joe's van conversion shop. Suppose Joe takes plain vans and converts them into custom vans and can produce either fine or fancy vans. The decision modeled is how many of each van type to convert this week. The number converted this week by van type constitutes the decision variables. We denote these variables as X_{fine} and X_{fancy}. Both types require a $25,000 plain van. Fancy vans sell for $37,000 and Joe uses $10,000 in parts to customize them yielding a profit margin of $2,000. Fine vans use $6,000 in parts and sell for $32,700 yielding profits of $1,700. Joe figures the shop can work on no more than 12 vans in a week. Joe hires 7 people including himself and operates 8 hours per day, 5 days a week and thus has at most 280 hours of labor available in a week. Joe also estimates that a fancy van will take 25 hours of labor, while a fine van will take 20 hours.

In order to set up Joe's problem as an LP, we must mathematically express the objective and constraint functions. Since the estimated profit per fancy vans is $2,000 per van, then $2{,}000X_{fancy}$ is the total profit from all the fancy vans produced. Similarly, $1{,}700\ X_{fine}$ is the total profit from fine van production. The total profit from all van conversions is $2{,}000\ X_{fancy} + 1{,}700\ X_{fine}$. This equation mathematically describes the total profit consequences of Joe's choice of the decision variables. Given that Joe wishes to maximize total profit, his objective is to determine the levels of the decision variables that

$$\text{Maximize } Z = 2000\ X_{fancy} + 1700\ X_{fine}$$

This is the objective function of the LP model.

Joe's factory has limited amounts of capacity and labor. In this case, capacity and labor are resources which limit the allowable (also called <u>feasible</u>) values of the decision variables. Since the decision variables are defined in terms of vans converted in a week, the total vans converted is $X_{fancy} + X_{fine}$. This sum must be less than or equal to the capacity available (12). Similarly, total labor use is given by $25\ X_{fancy} + 20\ X_{fine}$ which must be less than or equal to the labor available (280). These two limits are called constraints. Finally, it makes no sense to convert a negative number of vans of either type; thus, X_{fancy} and X_{fine} are restricted to be greater than or equal to zero. Putting it all together, the LP model of Joe's problem is to choose the values of X_{fancy} and X_{fine} so as to:

$$
\begin{array}{llllll}
\text{Maximize } Z = & 2000\,X_{fancy} & + & 1700\,X_{fine} & & \\
\text{s.t.} & X_{fancy} & + & X_{fine} & \leq & 12 \\
 & 25\,X_{fancy} & + & 20\,X_{fine} & \leq & 280 \\
 & X_{fancy} & , & X_{fine} & \geq & 0
\end{array}
$$

This is a formulation of Joe's LP problem depicting the decision to be made (i.e. the choice of X_{fancy} and X_{fine}). The formulation also identifies the rules, commonly called constraints, by which the decision is made and the objective which is pursued in setting the decision variables.

2.3 Other Forms of the LP Problem

Not all LP problems will naturally correspond to the above form. Other legitimate representations of LP models are:

1) Objectives which involves minimize instead of maximize i.e.,

$$\text{Minimize } Z = c_1 X_1 + c_2 X_2 + ... + c_n X_n.$$

2) Constraints which are "greater than or equal to" instead of "less than or equal to"; i.e.,

$$a_{i1} X_1 + a_{i2} X_2 + ... + a_{in} X_n \geq b_1.$$

3) Constraints which are strict equalities; i.e.,

$$a_{i1} X_1 + a_{i2} X_2 + ... + a_{in} X_n = b_1.$$

4) Variables without non-negativity restriction i.e., X_j can be unrestricted in sign i.e.,

$$X <=> 0$$

5) Variables required to be non-positive i.e., $X_j \leq 0$.

2.4 Assumptions of LP

LP problems embody seven important assumptions relative to the problem being modeled. The first three involve the appropriateness of the formulation; the last four the mathematical relationships within the model.

2.4.1 Objective Function Appropriateness

This assumption means that within the formulation the objective function is the sole criteria for choosing among the feasible values of the decision variables. Satisfaction of this assumption can often be difficult as, for example, Joe might base his van conversion plan not only on profit but also on risk exposure, availability of vacation time, etc. The risk modeling and multi-objective chapters cover the relaxation of this assumption.

2.4.2 Decision Variable Appropriateness

A key assumption is that the specification of the decision variables is appropriate. This assumption requires that

a) The decision variables are all fully manipulatable within the feasible region and are under the control of the decision maker.

b) All appropriate decision variables have been included in the model.

The nature and relaxation of sub-assumption (a) is discussed in the Advanced modeling considerations chapter in the "Common Mistakes" section, as is sub-assumption (b). Sub-assumption a) is also highlighted in Chapters IX and XVI.

2.4.3 Constraint Appropriateness

The third appropriateness assumption involves the constraints. Again, this is best expressed by identifying sub-assumptions:

a) The constraints fully identify the bounds placed on the decision variables by resource availability, technology, the external environment, etc. Thus, any choice of the decision variables, which simultaneously satisfies all the constraints, is admissible.

b) The resources used and/or supplied within any single constraint are homogeneous items that can be used or supplied by any decision variable appearing in that constraint.

c) Constraints have not been imposed which improperly eliminate admissible values of the decision variables.

d) The constraints are inviolate. No considerations involving model variables other than those included in the model can lead to the relaxation of the constraints.

Relaxations and/or the implications of violating these assumptions are discussed throughout the text.

2.4.4 Proportionality

Variables in LP models are assumed to exhibit proportionality. Proportionality deals with the contribution per unit of each decision variable to the objective function. This contribution is assumed constant and independent of the variable level. Similarly, the use of each resource per unit of each decision variable is assumed constant and independent of variable level. There are no economies of scale.

For example, in the general LP problem, the net return per unit of X_j produced is c_j. If the solution uses one unit of X_j, then c_j units of return are earned, and if 100 units are produced, then returns are $100c_j$. Under this assumption, the total contribution of X_j to the objective function is always proportional to its level.

This assumption also applies to resource usage within the constraints. Joe's labor requirement for fine vans was 25 hours/van. If Joe converts one fine van he uses 25 hours of labor. If he converts 10 fine vans he uses 250 hours (25*10). Total labor use from van conversion is always strictly proportional to the level of vans produced.

Economists encounter several types of problems in which the proportionality assumption is grossly violated. In some contexts, product price depends upon the level of production. Thus, the contribution per unit of an activity varies with the level of the activity. Methods to relax the proportionality assumption are discussed in the nonlinear approximations, price endogenous, and risk chapters. Another case occurs when fixed costs are to be modeled. Suppose there is a fixed cost associated with a variable having any non-zero value (i.e., a construction cost). In this case, total cost per unit of production is not constant. The integer programming chapter discusses relaxation of this assumption.

2.4.5 Additivity

Additivity deals with the relationships among the decision variables. Simply put their contributions to an equation must be additive. The total value of the objective function equals the sum of the contributions of each variable to the objective function. Similarly, total resource use is the sum of the resource use of each variable. This requirement rules out the possibility that interaction or multiplicative terms appear in the objective function or the constraints.

For example, in Joe's van problem, the value of the objective function is 2,000 times the fancy vans converted plus 1,700 times the fine vans converted. Converting fancy vans does not alter the per van net margin of fine vans and vice versa. Similarly, total labor use is the sum of the hours of labor required to convert fancy vans and the hours of labor used to convert fine vans. Making a lot of one van does not alter the labor requirement for making the other.

In the general LP formulation, when considering variables X_j and X_k, the value of the objective function must always equal c_j times X_j plus c_k times X_k. Using X_j does not affect the per unit net return of X_k and vice versa. Similarly, total resource use of resource I is the sum of $a_{ij}X_j$ and $a_{ik}X_k$. Using X_j does not alter the resource requirement of X_k. The nonlinear approximation, price endogenous and risk chapters present methods of relaxing this assumption.

2.4.6 Divisibility

The problem formulation assumes that all decision variables can take on any non-negative value including fractional ones; (i.e., the decision variables are continuous). In the Joe's van shop example, this means that fractional vans can be converted; e.g., Joe could convert 11.2 fancy vans and 0.8 fine vans.

This assumption is violated when non-integer values of certain decision variables make little sense. A decision variable may correspond to the purchase of a tractor or the construction of a building where it is clear that the variable must take on integer values. In this case, it is appropriate to use integer programming.

2.4.7 Certainty

The certainty assumption requires that the parameters c_j, b_i, and a_{ij} be known constants. The optimum solution derived is predicated on perfect knowledge of all the parameter values. Since all exogenous factors are assumed to be known and fixed, LP models are sometimes called non-stochastic as contrasted with models explicitly dealing with stochastic factors. This assumption gives rise to the term "deterministic" analysis.

The exogenous parameters of a LP model are not usually known with certainty. In fact, they are usually estimated by statistical techniques. Thus, after developing a LP model, it is often useful to conduct sensitivity analysis by varying one of the exogenous parameters and observing the sensitivity of the optimal solution to that variation. For example, in the van shop problem the net return per fancy van is $2,000, but this value depends upon the van cost, the cost of materials and the sale price all of which could be random variables.

Considerable research has been directed toward incorporating uncertainty into programming models. We devote a chapter to that topic.