Nonlinearities and Approximations

Minimization of the Sum of Absolute Deviations - transformation

1 

2. A constraint set is formed by moving the ∑ Xjibj term to the left side of the equation.



3. The basic problem of minimizing the summed absolute values of all €i is:



4. Conversion to LP problem: 

5. The resultant problem is:



Which is still nonlinear

Nonlinearities and Approximations

Minimization of the Sum of Absolute Deviations

- transformation

1.  which can be written as:



1. The formula then becomes:



This is an LP formulation except for the constraint on the product of €i+ and €i-. However, this constraint can be dropped.

Nonlinearities and Approximations

Minimization of the Sum of Absolute Deviations

- transformation

Reduced Problem:



Rearranging first constraint: 

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Y = 4 | | | Y = -6 | | |
| €+ | €- | €+  + € - | €+ | €- | €+  + €- |
| 4 | 0 | 4\* | 0 | 6 | 6\* |
| 16 | 12 | 28 | 14 | 20 | 34 |
| Z + 4 | Z | 2Z + 4 | Z | Z + 6 | 2Z + 6 |
| \* These cases are the only ones in which €+ \* €-equals zero. | | | | | |

Nonlinearities and Approximations

Minimization of the Sum of Absolute Deviations

- transformation An Example:

|  |  |  |
| --- | --- | --- |
| Price of Raw Oranges | Quantity of Oranges Sold | Quantity of Juice Sold |
| 10 | 8 | 5 |
| 5 | 9 | 1 |
| 4 | 10 | 9 |
| 2 | 13 | 8 |
| 6 | 15 | 2 |
| 9 | 17 | 3 |



An Alternative Formulation would be:



Nonlinearities and Approximations

Minimization of the Sum of Absolute Deviations

- transformation An Example:

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **Table 9.1. Minimization of Sum of Absolute Deviations Formulation** | | | | | | | | | | | | | | | |
|  | €1+ | €1- | €2+ | €2- | €3+ | €3- | €4+ | €4- | €5+ | €5- | €6+ | €6- | b0 | b1 | b2 |  |
| Obj | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |  |  |  | Min |
| 1 | 1 | -1 |  |  |  |  |  |  |  |  |  |  | 1 | 8 | 5 | =10 |
| 2 |  |  | 1 | -1 |  |  |  |  |  |  |  |  | 1 | 9 | 1 | =5 |
| 3 |  |  |  |  | 1 | -1 |  |  |  |  |  |  | 1 | 10 | 9 | =4 |
| 4 |  |  |  |  |  |  | 1 | -1 |  |  |  |  | 1 | 13 | 8 | =2 |
| 5 |  |  |  |  |  |  |  |  | 1 | -1 |  |  | 1 | 15 | 2 | =6 |
| 6 |  |  |  |  |  |  |  |  |  |  | 1 | -1 | 1 | 17 | 3 | =9 |

Nonlinearities and Approximations

Minimization of the Sum of Absolute Deviations

- transformation An Example:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Table 9.2. Solution of Minimization of Absolute Deviation Sum Example** | | | | | |
| Objective function = 11.277 | | | | | |
| Variable | Value | Reduced Cost | Equation | Slack | Shadow Price |
| 1+ | 5.787 | 0 | Obs 1 | 0 | 1 |
| 1- | 0 | 2.000 | Obs 2 | 0 | -0.660 |
| €2+ | 0 | 1.66 | Obs 3 | 0 | 0.191 |
| €1- | 0 | 0.340 | Obs 4 | 0 | -1 |
| €3+ | 0 | 0.809 | Obs 5 | 0 | -0.532 |
| €3- | 0 | 1.191 | Obs 6 | 0 | 1 |
| €4+ | 0 | 2.000 |  |  |  |
| €4- | 2.723 | 0 |  |  |  |
| €5+ | 0 | 1.532 |  |  |  |
| €5- | 0 | 0.468 |  |  |  |
| €6+ | 2.766 | 0 |  |  |  |
| €6- | 0 | 2.000 |  |  |  |
| b0 | 3.426 | 0 |  |  |  |
| b1 | 0.191 | 0 |  |  |  |
| b2 | -0.149 | 0 |  |  |  |

Nonlinearities and Approximations

Minimization of Largest Absolute Deviation

- transformation



Min e



The Composite Linear Program is:



Nonlinearities and Approximations

Minimization of Largest Absolute Deviation

- transformation Example:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Rows | € | b0 | b1 | b2 |  | |
| Objective | 1 |  |  |  | Minimize | |
| 1+ | -1 | -1 | -8 | -5 | ≤ | -10 |
| 1- | -1 | 1 | 8 | 5 | ≤ | 10 |
| 2+ | -1 | -1 | -9 | -1 | ≤ | -5 |
| 2-- | -1 | 1 | 9 | 1 | ≤ | 5 |
| 3+ | -1 | -1 | -10 | -9 | ≤ | -4 |
| 3- | -1 | 1 | 10 | 9 | ≤ | 4 |
| 4+ | -1 | -1 | -13 | -8 | ≤ | -2 |
| 4- | -1 | 1 | 13 | 8 | ≤ | 2 |
| 5+ | -1 | -1 | -15 | -2 | ≤ | -6 |
| 5- | -1 | 1 | 15 | 2 | ≤ | 6 |
| 6+ | -1 | -1 | -17 | -3 | ≤ | -9 |
| 6- | -1 | 1 | 17 | 3 | ≤ | 9 |
|  |  |  | 1 |  | ≤ | 0 |
|  |  |  |  | 1 | ≥ | 0 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Table 9.3. Solution of Largest Absolute Deviation Example** | | | | | |
| Variables | Value | Reduced Cost | Equation | Slack | Shadow Price |
| € | 3.722 | 0 | 1+ | 0 | -0.222 |
| b0 | 7.167 | 0 | 1- | 7.44 | 0.0 |
| b1 | -0.111 | 0 | 2+ | 4.89 | 0.0 |
| b2 | 0.000 | 2.056 | 2- | 2.56 | 0.0 |
|  |  |  | 3+ | 5.78 | 0.0 |
|  |  |  | 3- | 1.67 | 0.0 |
|  |  |  | 4+ | 7.44 | 0.0 |
|  |  |  | 4- | 0 | -0.5 |
|  |  |  | 5+ | 3.22 | 0.0 |
|  |  |  | 5- | 4.22 | 0.0 |
|  |  |  | 6+ | 0 | -0.278 |
|  |  |  | 6- | 7.44 | 0.0 |

Nonlinearities and Approximations

Optimizing a Fraction

- transformation



where: 

Transformation to LP:









Nonlinearities and Approximations

Optimizing a Fraction

- transformation

More Transformations:

Let yj = Xj y0







Nonlinearities and Approximations

Optimizing a Fraction / Example

- transformation

Suppose that it is desirable to solve the following problem



Then the transformed problem is



Once a solution to this problem is obtained, the values of the original variables are recovered using the formulas

X1 = y1 / y0

X2 = y2 / y0

Nonlinearities and Approximations

Optimizing a Fraction / Example

- transformation

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Table 9.4. Solution to the Example for Optimizing a Fraction** | | | | | |
| Objective function = 0.2899 | | | | | |
| Variable | Value | Reduced Cost | Equation | Slack | Shadow Price |
| y0 | 0.032 | 0 | 1 | 0 | 0.342 |
| y1 | 0.042 | 0 | 2 | 0 | 0.042 |
| y2 | 0.126 | 0 | 3 | 0 | 0.290 |

Nonlinearities and Approximations

Grid Point Approximations- Separable Programming -













Nonlinearities and Approximations

Separable Programming / Example 1

Suppose we approximate the problem.



To set this problem up, suppose we use values of X equal to 1,2,3,4,5,6 and the same values for Z. The separable programming representation is in Table 12.

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | λ1 | λ2 | λ3 | λ4 | λ5 | λ6 | Y | β 1 | β 2 | β 3 | β 4 | β 5 | β 6 |  |
| max | 3.75 | 7 | 9.75 | 12 | 13.75 | 13 |  | 1.25 | 3 | 5.25 | 8 | 10.125 | 15 |  |
| constraint 1 | 1 | 2 | 3 | 4 | 5 | 6 | -3 |  |  |  |  |  |  | ≤0 |
| constraint 2 |  |  |  |  |  |  | 2 | -1 | -2 | -3 | -4 | -5 | -6 | ≤0 |
| convex 1 | 1 | 1 | 1 | 1 | 1 | 1 |  |  |  |  |  |  |  | ≤1 |
| convex 2 |  |  |  |  |  |  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | ≤1 |

Note that stands for the amount of the gridpoint X=2 utilized having an objective value equal to the nonlinear function of X evaluated at X=2. The GAMS formulation is called SEPARABL and the resultant solution is shown in Table 9.5. The objective function value is 7.625. The model sets == 0.5 amounting to 50% of gridpoint X4 and 50% of X5 or X=4.5. The value of Y = 1.5. Simultaneously β1 = 1 implying Z = 3.

Solution:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Table 9.5. Solution to the Step Approximation Example** | | | | | |
| Objective function = 7.625 | | | | | |
| Variable | Value | Reduced Cost | Equation | Slack | Shadow Price |
| λ1 | 0 | -3.000 | 1 | 0 | 1.750 |
| λ2 | 0 | -1.500 | 2 | 0 | 2.625 |
| λ3 | 0 | -0.500 | 3 | 0 | 5.000 |
| λ4 | 0.5 | 0 | 4 | 0 | 2.625 |
| λ5 | 0.5 | 0 |  |  |  |
| λ6 | 0 | -0.500 |  |  |  |
| Y | 1.5 | 0 |  |  |  |
| γ1 | 0 | -1.250 |  |  |  |
| γ 2 | 0 | -0.375 |  |  |  |
| γ 3 | 1 | 0 |  |  |  |
| γ 4 | 0 | -0.125 |  |  |  |
| γ 5 | 0 | -0.750 |  |  |  |
| γ 6 | 0 | -1.875 |  |  |  |

Nonlinearities and Approximations

Separable Programming / Example 2

Suppose we wish to approximate the following problem



Selecting a grid for Y of 0, 1, 2, 3, 4 and 5, the separable programming formulation becomes



Nonlinearities and Approximations

Separable Programming / Example 2

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Table 9.6. Solution to the Constraint Step Approximation Problem** | | | | | |
| Objective function = 63.6 | | | | | |
| Variable | Value | Reduced Cost | Equation | Slack | Shadow Price |
| X | 23.2 | 0 | 1 | 0 | 3 |
| λ1 | 0 | -3.6 | 2 | 0 | 63.6 |
| λ2 | 0 | -1.2 |  |  |  |
| λ3 | 1 | 0 |  |  |  |
| λ4 | 0 | 0 |  |  |  |
| λ5 | 0 | -1.2 |  |  |  |
| λ6 | 0 | -3.6 |  |  |  |

Nonlinearities and Approximations

Functions of Multiple Variables



where there are multiple inputs and one output (for simplicity). The output X is a function of the levels of the multiple inputs (Yj). Also the function H(Y1...Yn) has to be such that this problem has a convex constraint set.

Nonlinearities and Approximations

Homogeneous of Degree One







Nonlinearities and Approximations

Homogeneous of Degree One / Example

Consider the problem:





|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Table 9.8. Solution to Example Problem for Homogeneous of Degree 1** | | | | | |
| Objective function = 719.8 | | | | | |
| Variable | Value | Reduced Cost | Equation | Slack | Shadow Price |
| X | 742.5 | 0 | 1 | 0 | 4 |
| α1 | 0 | -315.6 | 2 | 0 | 34.4 |
| α2 | 0 | -403.2 | 3 | 0 | 100 |
| α3 | 12.5 | 0 |  |  |  |
| Y1 | 50 | 14.4 |  |  |  |
| Y2 | 12.5 | 0 |  |  |  |

Nonlinearities and Approximations

Homogeneous of Degree Less Than One











The problem of interest becomes:



Nonlinearities and Approximations

Homogeneous of Degree Less Than One Example

# Consider the example problem



|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Table 9.9. Approximations for the Homogenous of Degree Less Than One Example** | | | | | | | | |
| X | Y11 | Y12 | X | Y12 | Y22 | X | Y13 | Y23 |
| 21 | 1 | 1 | 29.7 | 2 | 1 | 25.0 | 1 | 2 |
| 59.4 | 4 | 4 | 49.9 | 4 | 2 | 70.6 | 4 | 8 |
| 80.5 | 6 | 6 | 67.7 | 6 | 3 | 95.7 | 6 | 12 |
| 118.1 | 10 | 10 | 99.3 | 10 | 5 | 140.4 | 10 | 20 |

Nonlinearities and Approximations

Homogeneous of Degree Less Than One

Example

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **Table 9.10. Formulation of the Homogeneous Degree Less than One Example** | | | | | | | | | | | | | | | |
| Rows | X | λ11 | λ12 | λ13 | λ14 | λ21 | λ22 | λ23 | λ24 | λ31 | λ32 | λ33 | λ34 | Y1 | Y2 | RHS |
| Obj | 0.5 |  |  |  |  |  |  |  |  |  |  |  |  | -2 | -2 | max |
| x bal | 1 | -21 | -59.4 | -80.5 | -118.1 | -29.7 | -49.9 | -67.7 | -99.3 | -25.0 | -70.6 | -95.7 | -140.4 |  |  | = 0 |
| Y bal |  | 1 | 4 | 6 | 10 | 2 | 4 | 6 | 10 | 1 | 4 | 6 | 10 | -1 |  | = 0 |
|  |  | 1 | 4 | 6 | 10 | 1 | 2 | 3 | 5 | 2 | 8 | 12 | 20 |  | -1 | = 0 |
| convex |  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |  |  | ≤ 1 |
| Y lim |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 |  | ≤ 10 |

Nonlinearities and Approximations

Homogeneous of Degree Less Than One

Example

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Table 9.11. Solution to the Homogenous of Degree Less Than One Example** | | | | | |
| Objective function = 19.651 | | | | | |
| Variable | Value | Reduced Cost | Equation | Slack | Shadow Price |
| X | 99.3 | 0 | x bal | 0 | 0.500 |
| λ11 | 0 | -5.506 | Y1 bal | 0 | 2.850 |
| λ12 | 0 | -0.856 | Y2 bal | 0 | 2.000 |
| λ13 | 0 | 0.000 | convex | 0 | 11.156 |
| λ14 | 0 | -0.606 | Y lim | 0 | 0.85 |
| λ21 | 0 | -4.006 |  |  |  |
| λ22 | 0 | -1.581 |  |  |  |
| λ23 | 0 | -0.404 |  |  |  |
| λ24 | 1 | 0.000 |  |  |  |
| λ31 | 0 | -5.519 |  |  |  |
| λ32 | 0 | -3.237 |  |  |  |
| λ33 | 0 | -4.384 |  |  |  |
| λ34 | 0 | -9.434 |  |  |  |
| Y1 | 10 | 0.000 |  |  |  |
| Y2 | 5 | 0.000 |  |  |  |

Nonlinearities and Approximations

Iterative Approximations

Based on the concept of a Taylor series expansion. This method solves the problem :



The Approximating Problem:









