Primal and Dual LP Problems

Economic theory indicates that scarce (limited) resources have value. In LP models, limited resources are allocated, so they should be, valued.

Whenever we solve an LP problem, we implicitly solve two problems: the primal resource allocation problem, and the dual resource valuation problem.

Here we cover the resource valuation, or as it is commonly called, the Dual LP

Primal

 

Dual



Primal Dual Pair and Their Units

Primal

 

x is the variable and equals units sold

max sum (per unit profits) \* (units sold)

s.t. sum (per unit res. use)\*(units sold) < res on hand

 Maximizes profits subject to limited resources

Dual



## U is the variable and equals per unit resource value

min sum (per unit res value) \* (res on hand)

s.t. sum (per unit res value) \* (per unit res use)

 > per unit profits

Assigns value to the resources

Value of resources used in making a good must be at least as large and the value of that good

So resource values – shadow prices are set up so per unit profits are exhausted.

Primal Dual Pair Example

$\begin{matrix}Max&2000X\_{fancy}&+&1700X\_{fine}&+&1200X\_{new}&&&\\S.t.&X\_{fancy}&+&X\_{fine}&+&X\_{new}&\leq &12&(\frac{1}{2}days avail)\\&25X\_{fancy}&+&20X\_{fine}&+&19X\_{new}&\leq &280&(Labor)\\&X\_{fancy}&,&X\_{fine}&,&X\_{new}&\geq &0&Nonnegative\end{matrix}$

$$\begin{matrix}&(\frac{1}{2}days avail)&&(labor)&&&\\Min&12U\_{1}&+&280U\_{2}&&&\\S.t.&U\_{1}&+&25U\_{2}&\geq &2000&(X\_{fancy})\\&U\_{1}&+&20U\_{2}&\geq &1700&(X\_{fine})\\&U\_{1}&+&19U\_{2}&\geq &1200&(X\_{new})\\&U\_{1}&,&U\_{2}&\geq &0&Nonnegative\end{matrix}$$

Primal rows become dual columns

Primal Dual Correspondence

Fundamental Relations

Matrix Form

Primal Dual

Max CX min Ub

 AX < b UA > C

 X > 0 U **>** 0

 Dual Variable

Given Feasible X\* and U\* Relation

Primal Obj < Dual Obj to Partial

Derivative

CX\* < U\*b

 

Complementary Slackness Zero Profits

At Optimal X\*, u\* at Optimal

U\* (b - AX\*)=0 U\* b = C X\*

 (U\*A - C)X\*=0

Primal Dual Interrelations

## Relation Between Primal and Dual Objective functions

Suppose we have feasible primal (X\*) and dual (U\*) solutions. This means

 Max CX\* min U\*b

 AX\* < b U\*A > C

 X\* > 0 U\* > 0

Now given AX\*< b we can multiply through by U\* without changing inequality since U\* > 0 yielding

U\*AX\* < U\*b

Also given U\*A > C we can multiply through by X\* without changing inequality since X\* > 0

U\*AX\* > CX\*

Note here both of these have the term U\*AX\* and we can unify the two expressions

CX\* < U\*AX\* < U\*b or CX\* < U\*b

And thus given two feasible solutions the primal objective function is less than or equal to the dual objective function

Primal Dual Interrelations

## Is CBB-1 the Dual Solution

Note we can see if U\*=CBB-1 is a feasible and optimal dual solution.

Given optimal primal XB\* = B-1b and XNB\* = 0. This solution is also feasible;

XB\* = B-1b > 0 XNB = 0 so all X\* > 0

and to be optimal we have nonnegative reduced cost

CBB-1ANB - CNB > 0.

Given this, suppose we try = CBB-1 as a dual solution.

First, is this feasible in the dual constraints.

To be feasible, we must have U\*A > C and U\* > 0.

We know CBB-1ANB - CNB > 0 at optimality and is we replace CBB-1 with this is equivalent to the reduced cost criteria,

CBB-1ANB - CNB > 0 so for non basic variables U\*A > C

Further, for basic variables reduced cost is

CBB-1AB - CB

Where AB is the constraint coefficients of the basic variables and where that under rearrangement is B. So substituting B in place of AB the above equation becomes

CBB-1B - CB which is CB I - CB or CB - CB = 0

And this equals zero. Thus the reduced costs for basic variables is zero and we have U\*A > C so the dual constraints are satisfied.

Primal Dual Interrelations

## Is CBB-1 the Dual Solution

Now we need to see of the dual nonnegativity conditions U > 0 satisfied. We can look at this by looking at the implication of the nonnegative reduced costs for the slacks (UA>C).

For the slacks, suppose out original problem was

 Max C'X

 A'X < b

 X > 0

And we added slacks to become

 Max C'X + 0S

 A'X + IS < b

 X , S > 0

So in the problem A contains an identity matrix (A' I) and the associated entries in C are (C' O) so for the slacks (S) they are all 0's.

Thus for the slacks CBB-1As - Cs > 0 where As=I and Cs=0 yielding U\* I – 0 > 0 or U\* > 0.

So the U's are non-negative. Thus, U=CBB-1 satisfies all the dual constraints and is thus a feasible dual solution.

Primal Dual Interrelations

## Constructing Dual Solutions

Now the question becomes, is this choice optimal?

In this case the primal objective function Z equals CX\* which is CBXB\* + CNBXNB\* and then replacing XB\* with B-1b and XNB\* we get CBB-1b + CNB0 and this becomes CBB-1b. Then replacing CBB-1 with U\* we get U\*b which equals the dual objective function value.

So, the primal and dual objectives are equal CX\*=U\*b at U=CBB-1.

Finally since we proved for any feasible X0, U0 that the primal objective is less than or equal to the dual one CX0 < U0b and they are now equal this must be optimal.

The dual objective can get no smaller and the primal no larger so we are optimal for both and U\*=CBB-1 is the optimal solution to the dual problem.

This demonstration shows that given the solution from the primal the dual solution can simply be computed as CBB-1 without need to solve the dual problem. Furthermore these are the shadow prices or in GAMS the equation marginals that are reported in every solution.

This also shows when resources are valued at U\* there are no profits ie all profits are allocated to resources.

Primal Dual Interrelations

## Complementary slackness

Above we showed two things

For feasible X\*, U\*

CX\* < U\*AX\* < U\*b

and for the choice U=CBB-1 where B is the optimal basis.

CX\*=U\*b

So this means

CX\* = U\*AX\* = U\*b

Now let us deal with this as two expressions

CX\* = U\*AX\* or U\*AX\* - CX\* = 0 or (U\*A - C)X\* = 0

and

U\*AX\*=U\*b or U\*AX\*- U\*b=0 or U\*(AX\* - b)=0

So we get

(U\*A - C) X\* = 0

U\* (AX\* - b) = 0

These conditions are known as complementary slackness.

Primal Dual Interrelations

## Complementary slackness

Given (U\*A - C) X\* = 0

U\* (AX\* - b) = 0

Along with X > 0 (primal non neg)

 UA > C or UA-C > 0 (dual constraint)

 U > 0 (dual non neg)

 AX < b or AX-b < 0 (primal constraint)

This means both parts of (U\*A - C) X\* = 0 are positive and this can only be satisfied when every term is zero.

Thus when U\*A>C then X\*=0 and when X\*>0 then U\*A=C. So when variables have non zero reduced costs they equal zero and when reduced costs are non zero then variables are zero.

Also in U\*(AX\*-b) the term can only take on negative or zero values and to equal 0 the every term in (U\*A - C) X\* must be zero.

So when constraints are binding (AX=b) then shadow princes are positive (or zero) and when constraints are loose (AX<b) then shadow prices must be zero.

These are the complementary relationships

Primal Dual Pair Example

Primal $\begin{matrix}Max&2000X\_{fancy}&+&1700X\_{fine}&+&1200X\_{new}&&&\\S.t.&X\_{fancy}&+&X\_{fine}&+&X\_{new}&\leq &12&(\frac{1}{2}days avail)\\&25X\_{fancy}&+&20X\_{fine}&+&19X\_{new}&\leq &280&(Labor)\\&X\_{fancy}&,&X\_{fine}&,&X\_{new}&\geq &0&Nonnegative\end{matrix}$

Solution

**Objective : 22800.000000**

**---- EQU resavail LOWER LEVEL UPPER MARGINAL**

**capacity -INF 12.0000 12.0000 500.0000**

**labor -INF 280.0000 280.0000 60.0000**

**- VAR x LOWER LEVEL UPPER MARGINAL**

**X\_fancy . 8.0000 +INF .**

**X\_fine . 4.0000 +INF .**

**X\_new . . +INF -440.0000**

Dual$\begin{matrix}&(\frac{1}{2}days avail)&&(labor)&&&\\Min&12U\_{1}&+&280U\_{2}&&&\\S.t.&U\_{1}&+&25U\_{2}&\geq &2000&(X\_{fancy})\\&U\_{1}&+&20U\_{2}&\geq &1700&(X\_{fine})\\&U\_{1}&+&19U\_{2}&\geq &1200&(X\_{new})\\&U\_{1}&,&U\_{2}&\geq &0&Nonnegative\end{matrix}$

Solution

**Objective : 22800.000000**

**---- EQU redcost LOWER LEVEL UPPER MARGINAL**

**X\_fancy 2000.0000 2000.0000 +INF 8.0000**

**X\_fine 1700.0000 1700.0000 +INF 4.0000**

**X\_new 1200.0000 1640.0000 +INF .**

 **(note diff is 440)**

**---- VAR u LOWER LEVEL UPPER MARGINAL**

**capacity . 500.0000 +INF .**

**labor . 60.0000 +INF .**

Primal Dual Interrelations

Primal Dual Solution Correspondence

|  |  |
| --- | --- |
| Primal Solution ItemPrimal Solution Information | Dual Solution ItemCorresponding Dual Solution Information |
| Objective function | Objective function |
| Shadow prices | Variable values |
| Slacks | Reduced costs |
| Variable values | Shadow prices |
| Reduced costs | Slacks |

Primal Dual Interrelations

## Interpreting Dual Solutions

In addition given the derivation in the last chapter we can establish the interpretation of the dual variables. In particular, since the optimal dual variables equal CB B-1 (which are called the primal shadow prices) then the dual variables are interpretable as the marginal value product of the resources since we showed



Also

Complementary Slackness Zero Profits

At Optimal X\*, U\* Given Optimal

 U\*'b = cx\*

**Duals of other model forms**

Above [primal](http://agrinet.tamu.edu/mccarl/gms/ch04/PRIMAL2.GMS) problem has been Max subject to less thens and nonnegativity. There given a LP problem of the form in the left its dual is of the form on the right

Primal Dual

Max CX min Ub

 AX < b UA > C

 X > 0 U > 0

But what is the primal looks like

Max CX +eY

 AX +DY < b

 EX + MY > f

 GX+PY = h

 X > 0 , Y unrestricted

Well in general the form of the primal constraints determines the restrictions on the sign of the associated dual variable. And the sign restriction on variables in the primal determines the form of the constraint inequality or equality.

You can either memorize some things or convert all to standard for. Here we do the later

**Duals of other model forms**

Given Max CX +eY

 AX +DY < b

 EX + MY > f

 GX+PY = h

 X > 0 , Y unresticted

We can write

 EX + MY > f as -EX - MY< -f

and

 GX+PY = h as two constraints

GX+PY < h and GX+PY> h or –GX-PY< - h

Then for the variable Y we can replace it with two nonnegative variables Y=Y1-Y2

So now our problem is

Max CX + eY1 – eY2

 AX + DY1 - DY2 < b

 -EX – MY1 + MY2 < - f

 GX + PY1 - PY2 < h

 -GX - PY1 + PY2 < - h

 X , Y1 , Y2 > 0

**Duals of other model forms**

So now our problem is

Max CX + eY1 – eY2

 AX + DY1 - DY2 < b

 -EX – MY1 + MY2 < - f

 GX + PY1 - PY2 < h

 -GX - PY1 + PY2 < - h

 X , Y1 , Y2 > 0

and the dual is

Min U1b - U2f + U3h – U4h

 U1A - U2E + U3G – U4G > C

 U1D - U2M + U3P – U4P > e

 -U1D + U2M - U3P + U4P > - e

 U1 , U2, U3 , U4 > 0

and if we recognize

 U1D - U2M + U3P – U4P > e

 -U1D + U2M - U3P + U4P > - e

Means U1D - U2M + U3P – U4P = e

And we can say U=U3-U4

The dual becomes

Min U1b - U2f + Uh

 U1A - U2E + UG > C

 U1D - U2M + UP = e

 U1 , U2, > 0 U unrestricted

There are some relations on this covered in the book in Table 4.1

Degeneracy and Duality

The above interpretations for the dual variables depend upon whether the basis still exists after the change occurs.

When a basic primal variable equals zero, dual has alternative optimal solutions. The cause of this situation is generally primal constraints are redundant at the solution point and the range of right hand sides is zero.



At the optimal solution, X1 = 50, X2 = 50, constraints are redundant.

If first slack variable is basic then X1 = 50, X2 = 50, S1 = 0 while S1 is basic. Shadow prices are 0, 3, and 2.

If S3 basic X1 = 50, X2 = 50 S3 = 0 with shadow prices 2, 1, 0. Same objective value -- multiple solutions.

Degeneracy and Duality



With solutions [u1 u2 u3] = [0 3 2] or [ 2 1 0 ]

The main difficulty with [degeneracy](http://agrinet.tamu.edu/mccarl/gms/ch04/Degen.GMS) is in interpreting the shadow prices as they take on a direction.

If one were to increase the first right hand side from 100 to 101 this would lead to a zero change in the objective function and X1 and X2 would remain at 50.

Decrease first constraint rhs from 100 to 99 then objective function which is two units smaller because X2 would need to be reduced from 50 to 49.

This shows that the two alternative shadow prices for the first constraint (i.e., 0 and 2) each hold in a direction.

Similarly if bound on X1 51, obj increases by 1, whereas, if moved downward to 49, it would cost 3.

Meanwhile, reducing X2 bound costs 2 and increasing by 0. this explains all shadow prices

Primal Columns are Dual Constraints

Columns in the primal, form constraints on the dual shadow price information.

Thus, for example, when a column is entered into a model indicating as much of a resource can be purchased at a fixed price as one wants, then this column forms an upper bound on the shadow price of that resource.

Note that it would not be sensible to have a shadow price of that resource above the purchase price since one could purchase more of that resource.

Similarly, allowing goods to be sold at a particular price without restriction provides a lower bound on the shadow price.

In general, the structure of the columns in a primal linear programming model should be examined to see what constraints they place upon the dual information.

The linear programming modeling chapter extends this discussion.