**Solving and interpreting LP Models**

**Slack Variables**

**Given Problem**

Max C1X1

s.t. A1X1 < b

X1 > 0

**Add Slacks**

Max C1X1 + 0S

s.t. A1X1 + IS = b

X1 , S > 0

**From now on**

Max CX

s.t. AX = b

X > 0

Where C= (C1 0) , X=(X1 S), A=(A1 I)

**Solving and interpreting LP Models**

**Matrix Solution**

Mathematicians have found the above can be solved by choosing M variables to be nonzero and setting the rest to zero then inverting. The set of variables chosen for the set of M are called basic. The remaining variables are called non basic. So we partition the problem

  MAX CBXB + CNBXNB

s.t. BXB + ANBXNB = b

XB , XNB > 0.

where XB are the basic variables, and

XNB are the non-basic variables.

So

BXB + ANBXNB = b

Andsetting XNB to zero

BXB + 0 = b

**Or**

XB = B-1 b

**Solving and interpreting LP Models**

**Matrix Solution**

 And if we consider non basic

XB = B-1 b - B-1ANBXNB

Now suppose we don’t know if we have the right basis

Our objective function is

Z = CBXB + CNBXNB

 Substituting in the XB equation yields

Z = CB(B-1b – B-1ANBXNB) + CNBXNB

or

Z = CBB-1b - CBB-1ANBXNB + CNBXNB

or

Z = CBB-1b - (CBB-1ANB - CNB)XNB

**Solving and interpreting LP Models**

**Matrix Solution**

Now we know

Z = CBB-1b - (CBB-1ANB - CNB)XNB

Or



The current incumbent value when Xj is set to zero is

Z = CBB-1 b

But note



So if we have a basis and want to increase the objective we would choose to add a variable with a negative term

When CBB-1Aj - Cj

is all positive we are optimal

**Solving and interpreting LP Models**

**Matrix Solution**

So we want to increase non basic variable η. We know we need to maintain XB > 0. So using our above formulae when we increase Xη which is a member of the non basic vector but leaving other X's at zero the altered values are

XB = B-1 b - B-1ANBXNB

When using just Xη and recalling the basic variables must remain non negative

XBi = (B-1 b) i - (B-1Aη) iXη > 0

An element of the basis reaches zero when

XBi = (B-1 b) i - (B-1Aη) iXη = 0 for some i

Since all XBi must remain positive or zero

Xη < (B-1 b) i /(B-1Aη) i for all i

when (B-1Aη) i > 0

and to find first one to zero or the biggest possible value of Xη we have the so called minimum ratio rule

Xη = min ( (B-1 b) i /(B-1Aη) i ) over i

when (B-1Aη) i > 0

**Solving and interpreting LP Models**

**Simplex Method**

1) Select an initial feasible basis matrix (B).

2) Calculate the Basis inverse (B-1).

3) Calculate CBB-1aj - cj for the non-basic variables.

Identify the entering variable as one with the most negative value of that calculation;

if there are none, go to step 6.

4) Calculate the minimum ratio rule.

X**η** = min ( (B-1 b) i /(B-1A**η)** i ) over i

when (B-1A**η)** i **> 0**

Denote row with minimum ratio as row i\*;

If there are no rows with (B-1A**η)** i **> 0** then go to step 7.

5) Replace the variable basic in row i\* with variable **η** recalculate the basis inverse. Go to step 3.

6) The solution is optimal. Terminate.

Optimal basic variable values = B-1b

Non-basic variables = 0

Reduced costs = CBB-1aj - cj Objective function = CBB-1b.

7) The problem is unbounded. Terminate.

**Solving and interpreting LP Models**

**Simplex Example**

**Problem**

Maximize Z = 2000 Xfancy + 1700 Xfine

s.t. Xfancy + Xfine  12

25 Xfancy + 20 Xfine  280

Xfancy , Xfine  0

**Add Slacks**

Maximize Z = 2000Xfancy +1700Xfine + 0 S1 + 0 S2

s.t. Xfancy+ Xfine + S1  = 12

25Xfancy+ 20 Xfine + S2 = 280

Xfancy , Xfine , S1 ,  S2   0

**Matrix Setup**

C = [ 2000 1700 0 0 ] A= 1 1 1 0

25 20 0 1

b= 12

280

**Initial Basis**

XB = [S1 S2] XNB = [Xfancy Xfine] = 0 0

B = 1 0 ANB= 1 1 CB = [ 0 0 ]

0 1 25 20 CNB = [2000 1700]

B-1 = 1 0 CB B-1 ANB - CNB= [ -2000 -1700 ]

0 1

XB =B-1b = 12 CB B-1 b = 0

280

**Entering Variable**

Xfancy

**Solving and interpreting LP Models**

**Simplex Example (continued)**

**Example Continued**

**Min ratio rule**

**XB** = B-1b - B-1Axfancy **X**fancy= 12 - 1 **X**fancy > 0

280 25

**Or X**fancy < 12/1 = 12/1

280/25 11.2

**So take out second basic variable (S2) replace with Xfancy**

**Remove S2 Insert Xfancy**

**New Basis**

B = 1 1 ANB= 1 0 CB = [ 0 2000 ] CNB = [1700 0 ]

0 25 20 1

XB = [S1 Xfancy] XNB= [Xfine S2 ]

B-1 = 1 -1/25 CB B-1 ANB - CNB= [ -100 +80 ]

0 1/25

B-1b = 0.8 CB B-1 b = 22400

11.2

**Entering Variable**

Xfine

**Solving and interpreting LP Models**

**Which Variable Leaves**

**Min ratio rule**

**XB** = B-1b - B-1Axfancy **X**fine= 0.8 - 1/5 **X**fine > 0

11.2 4/5

**Or X**fine < (4/5)/(1/5) 4

(56/5)/(4/5) 14

**So take out first basic variable (S1) replace with Xfine**

**Remove S2 Insert Xfine**

**New Basis**

B = 1 1 ANB= 1 0 CB = [1700 2000 ] CNB = [0 0 ]

20 25 0 1

XB = [Xfine Xfancy] XNB= [S1 S2 ]

B-1 = 5 -1/5 CB B-1 ANB - CNB= [ 500 60 ]

-4 1/5

B-1b = 4 CB B-1 b = 22800

8

**Optimal Values**

Z = CBXB = 22800 CB B-1= [500 60]

Xfine=4 Xfancy=8

**Solving and interpreting LP Models**

**Sensitivity Information**

**Two Equations**





**Sensitivity Results**



For example

B-1 = 5 -1/5 CB B-1 ANB - CNB= [ 500 60 ]

-4 1/5

B-1 ANB= 5 -1/5 CB B-1 = [ 500 60 ]

-4 1/5

**Solving and interpreting LP Models**

**Right Hand Side Ranging**

**Two Fundamental Equations**



**Now Suppose We Alter RHS**

bnew = bold + θr

**Effect on Equations**



while CBB-1aj – cj is unchanged.

The net effect is that the new solution levels are equal to the old solution levels plus θB-1r. Objective function changes by this times CB

**Limits on** θ





**Solving and interpreting LP Models**

**Right Hand Side Ranging**

**Now Suppose We Alter RHS**

bnew = bold + θr

r= [1 0 ] (transposed)

**Effect on Equations**





**Limits on** θ





-4/5 ≤ θ ≤ 8/4

11.2 ≤ b1 ≤ 14**Solving and interpreting LP Models**

**Cost Ranging**

**Two Fundamental Equations**



**Now Suppose We Alter Obj**



**Effect on Equations**



while B-1b is unchanged.

The net effect is that the new reduced costs are equal to the old ones plus a term



Recalling this must be non zero we can solve for γ

In turn, we discover for non-basic variables



**Solving and interpreting LP Models**

**Cost Ranging**

while for basic variables



Example

Suppose in our example problem we want to alter the objective function on Xfancy so it equals 2,000 + γ. The setup then is



and



note the order that variables appear in the TB vector reflects the order they appear in the basis so Xfine is first then Xfancy

**Solving and interpreting LP Models**

**Cost Ranging**

So for the non-basic variables reduced cost equals

which implies -300< γ < 125 or that the basis is optimal for any objective function value for Xfancy between 2125 and 1700. This shows a range of prices for Xfancy for which optimal level is constant.

**Solving and interpreting LP Models**

**A1J Ranging**

Suppose we alter the matrix of the technical coefficients 

where , A, and M are mxn matrices. The matrix M indicates a set simultaneous of changes to be made in A. Then the expected change in the optimal objective function is



where u=CBB-1

**Example**



.

Thus, the change in the value of the objective function is given by



Suppose θ = -1, then the anticipated change ΔZ = -U\*MX\* = 720

Resolving revised problem shows objective function changes by 720.

**Solving and interpreting LP Models**

**Degeneracy**



The third constraint is redundant to the first two.

In the simplex solution X2 can be entered in place of second or third slack. If X2 is brought into the basis in the second row, the shadow prices determined are (u1, u2, u3) = (100, 75, 0).

If X2 is brought in the third row the value of the shadow prices are (u1, u2, u3) = (25, 0, 75).

These differ depending on whether the second or third slack variable is in the basis at a value of zero. Thus, the solution is degenerate



**Solving an interpreting LP Models**

**Alternative Optimal**



The objective is parallel to the binding constraints.

In the simplex solution X1 and X2 would have same reduced cost.

If X1 is brought into the basis, the solution is X1 = 50 & X2 = S1 =0 reduced costs = (0 , 5), shadow price = 5, obj = 250.

If X2 is brought into the basis, the solution is X2 = 50 & X1 = S1 =0 reduced costs = (0 , 5), shadow price = 5, obj = 250.

An alternative optimal with a non basic variable having zero reduced cost.

**Solving and interpreting LP Models**

**Shadow Prices and Bounded Variables**

Linear programming codes impose upper and lower bounds on individual variables in a special way and this influences shadow prices. An example of a problem with upper and lower bounds is given below.



The second constraint imposes an upper bound on X1, i.e., X1 < 10, while the third constraint, X2 > 1, is a lower bound on X2. Most LP algorithms allow one to specify these particular restrictions as either constraints or bounds. Solutions from LP codes under both are shown in Table 3.2.

**Solution with Bounds Imposed as Constraints**

Variable Value Reduced Equation Level Shadow

Cost Price

X1 10 0 1 4 0

X2 1 0 2 0 3

3 0 -1

**Solution with Bounds**

Variable Value Reduced Equation Level Shadow

Cost Price

X1 10 -3 1 4 0

X2 1 1

**Solving and interpreting LP Models**

**Artificial Variables**

These are variables added to the problem which permit an initial feasible basis, but need to be removed before solution is finalized.

**Max cx max cx +0 S1 +0\*s2-999a1-999a2**

**Ax < b ax +s1 =b**

**dx> f dx -s2 +a1 =f**

**gx=e gx +a2 =e**

**x>0 x, s1, s2, a1,a2 > 0**

**Problem**



Artificial variables are entered into each constraint which is not satisfied when X=0 and does not have an easily identified basic variable.

In this example, three sets of artificial variables are required.



Here A2, A3, and A4 are the artificial variables which permit an initial feasible nonnegative basis but which must be removed before a "true feasible solution" is present. Note that S1, A2, A3, and A4 can be put into the initial basis.

**Solving and interpreting LP Models**

**Artificial Variables**

Two approaches

1. phase 1 /phase 2

Initially minimize sum of A

2. Big M

Put -9999 \*A in objective

As follows



|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Table 3.4. Solution to the Big M Problem** | | | | |
| Variable | Value | Reduced Cost | Equation | Shadow Price |
| x1 | 1.333 | 0 | 1 | 1.667 |
| x2 | 4.333 | 0 | 2 | 0 |
| S1 | 0 | 1.667 | 3 | -1.333 |
| S2 | 1 | 0 | 4 | 0 |
| W | 4.667 |  |  |  |
| A2 | 0 | -99 |  |  |
| A3 | 0 | -97.667 |  |  |
| A4 | 0 | -99 |  |  |