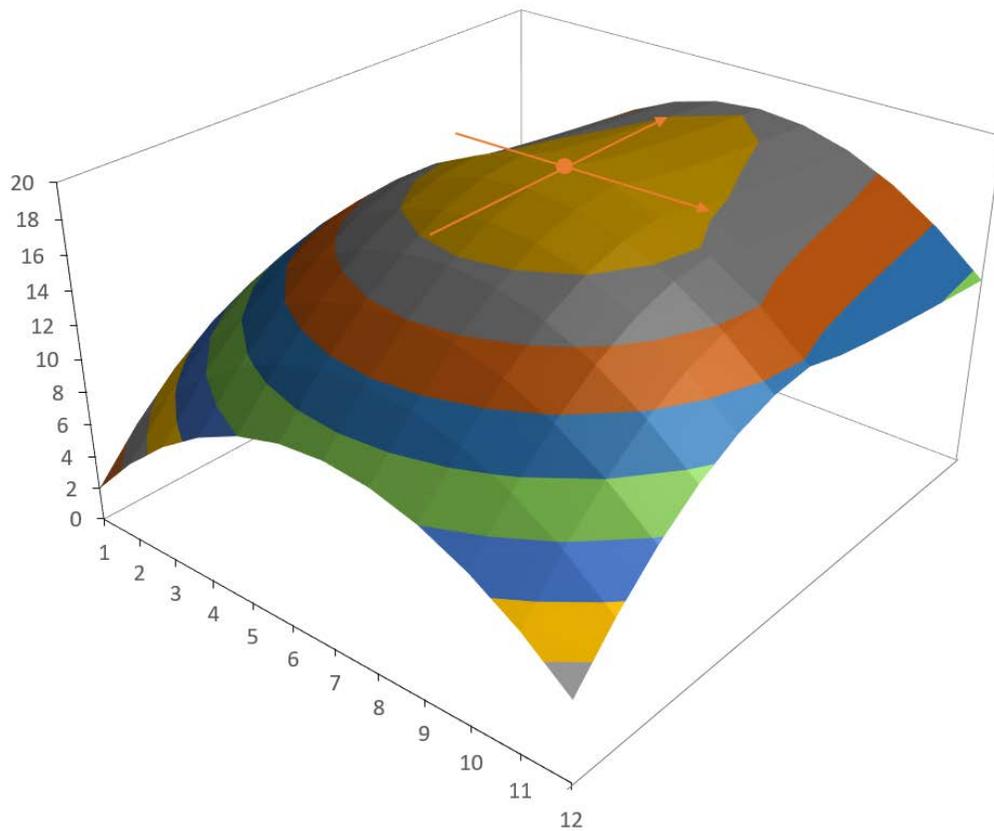# Applied Mathematical Programming

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Disclaimer

This book is intended to both serve as a reference guide and a text for a course on Applied Mathematical Programming for upper undergraduate and Master level students in Economics, Applied Economics, Agricultural and Resource Economics, and Management; primarily based on McCarl and Spreen (2013) _Applied Mathematical Programming Using Algebraic System_, Department of Agricultural Economics, Texas A&M University, College Station, TX.

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1. INTRODUCTION

This book is intended to both serve as a reference guide and a text for a course on Applied Mathematical Programming for upper undergraduate and Master level students in Economics, Applied Economics, Agricultural and Resource Economics, and Management; primarily based on McCarl and Spreen (2013)[1]. The material presented in McCarl and Spreen (2013) concentrates upon conceptual issues, problem formulation, computerized problem solution, and results interpretation; it is designed for the advanced readers who are familiar with mathematical economics including linear and matrix algebra and also with advanced modeling skills. Upper level undergraduate and/or Master students may not be beneficial from the book.

This booklet is intended to serve as an introductory guide and covers very basics of conceptual issues in mathematical programming and explores problem formulation with applied issues in decision making. Advanced topics including solution algorithms will be discussed only to the extent necessary to build the model and interpret solutions.

1.1. Mathematical Programming Approach

Mathematical programming (MP) refers to a set of procedures dealing with the analysis of optimization problems. Optimization problems are generally those in which a decision maker wishes to optimize some measure(s) of satisfaction (for example, profit) by selecting values for a set of variables (for example, production). We will discuss the set of mathematical programs where the variable values are constrained by conditions external to the problem at hand (for example, constraints on the maximum amount of resources available and/or the minimum amount of certain items which need to be on hand) and sign restrictions on the variables.

The general mathematical programming problem we will use is:

$$\text{Optimize } z = f(x)$$

(1-1) $\quad$ subject to $g(x) \in s_1$

$$x \in s_2$$

Here x is a <u>decision</u> variable. The level of x is chosen so that an objective is optimized. The objective is expressed algebraically as $z = f(x)$. The function $f(x)$ is commonly called the objective function and tells how alternative choices of x effect the decision maker satisfaction in terms of the objective. This objective function will be maximized or minimized. However, in setting x, a set of constraints must be obeyed requiring that the x's behave in some manner. These constraints are reflected in the above formulation by the requirements that: a) $g(x)$ must belong to s_1 and b) the variable must fall into s_2.

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[1] McCarl, Bruce .A., and Spreen, Thomas .H. (2013) *Applied Mathematical Programming Using Algebraic System*, Department of Agricultural Economics, Texas A&M University, College Station, TX; available at http://agecon2.tamu.edu/people/faculty/mccarl-bruce/books.htm

A number of applications have been cast into MP terms. Some examples of practical applications are

1. A firm wishes to minimize the cost of feeding cattle so sets up an LP problem. In this problem the objective is to minimize the cost of feeding expressed as the cost per lb of each ingredient times the amount of feed used summed over all feed stuff possibilities. The variables are the amount of each feedstuff used. However, in choosing the quantity of feedstuffs the diet must be structured so it meets the nutritional requirements of the animals. Thus, for example, constraints are needed insuring the calorie and protein content summed across all the feedstuffs used is greater than or equal to the animal requirement.

2. A firm wishes to learn how to manage its production facilities given that it may choose to either produce a good or buy it from another manufacturer and resell it. Specifically suppose as firm is in the business of electricity sale and can either generate it or buy it from a distant plant to meet customer needs. In such a case the model built would minimize the cost of generating or purchasing plus delivering energy given constraints on productive capacity, cost volume relationships, transmission capacity, demand and other factors. The variables would be quantity generated by facility, quantity purchased by supplier and quantity moved across the transmission lines.

3. A firm may wish to determine how to cut up a set of incoming logs to maximize profits. In such case the firm would introduce variables for the way to process the logs and the sale of final products. Constraints would be imposed on the quantity of logs by type, log handing facilities and product demand.

As the examples above illustrate, the MP problem encompasses many different types of problems some of which will be discussed in this book. In particular, if f(x) and g(x) are linear and the x is non-negative, then the problem becomes a <u>linear programming (LP) problem</u>. If the $x \in s_2$ restriction requires x to take on integer values, then this is an <u>integer programming problem</u>. If g(x) is linear, f(x) quadratic, and the s_2 restrictions are simply non-negativity restrictions, then we have a <u>quadratic programming problem</u>. Finally, if f(x) and g(x) are general nonlinear functions with s_2 being non-negativity conditions, the problem is a <u>nonlinear programming (NLP) problem</u>.

1.2. Mathematical Programming in Use

Mathematical programming (MP) is most often thought of as a technique which decision makers can use to develop optimal values of the decision variables. However, there are a considerable number of other potential usages of MP. Furthermore, as we will argue below, numerical usage for identification of specific decisions is probably the least common usage in terms of relative frequency.

Three sets of usages of MP that we regard as common are: 1) problem insight construction; 2) numerical usages which involve finding model solutions; and 3) solution algorithm development and investigation. We will discuss each of these in turn.

1.2.1. Generating Problem Insight

MP forces one to state a problem carefully. One must define: a) decision variables, b) constraints, c) objective function, d) linkages between variables and constraints that reflects complementary, supplementary, and competitive relationships among variables, and e) consistent data. The decision maker is forced to understand the problem interacting with the situation thoroughly, discovering relevant decision variables and constraining factors. Frequently, the resultant knowledge outweighs the value of any solutions and is probably the number one benefit of most mathematical programming exercises.

A second insight generating usage of MP involves analytical investigation of problems. While it is not generally acknowledged that MP is used, it provides the underlying basis for a large body of microeconomic theory. Often one sets up, for example, a utility function to be maximized subject to a budget constraint, then uses MP results for the characterization of optimal values. In turn, it is common to derive theoretical conclusions and state the assumptions under which those conclusions are valid. This is probably the second most common usage of MP and again is a non-numerical use.

1.2.2. Numerical Mathematical Programming

Numerical usages fall into four subclasses: a) prescription of solutions, b) prediction of consequences, c) demonstration of sensitivity, and d) solution of systems of equations.

Prescription of solution: The most commonly thought of application of MP involves the prescriptive or normative question: Exactly what decision should be made given a particular specification of objectives, variables, and constraints? This is most often perceived as the usage of MP, but is probably the least common usage over the universe of models. In order to understand this assertion, one simply has to address the question: "Do you think that many decision makers yield decision making power to a model?" Very few circumstances entail this kind of trust. Most often, models are used for decision guidance or to predict the consequences of actions. One should adopt the philosophical position that models are an abstraction of reality and that an abstraction will yield a solution suggesting a practical solution, not always one that should be implemented.

Prediction: The second numerical MP usage involves prediction. Here the model is assumed to be an adequate depiction of the entity being represented and is used to predict in a conditional normative setting. Typically, this occurs in a business setting where the model is used to predict the consequences of environmental alterations (caused by investments, acquisition of resources, weather changes, market price conditions, etc.). Similarly, models are commonly used in government policy settings to predict the consequences of policy changes. Models have been used, for example, to analyze the implications for social benefits of a change in ambient air quality. Predictive use is probably the most common numerical usage of MP.

Demonstration of sensitivity: The third and next most common numerical usage of MP is sensitivity demonstration. Many research inquiries are of this nature where no one ever tries to implement the solutions, and no one ever uses the solutions for predictions. Rather, the model is used to

demonstrate what might happen if certain factors are changed. Here the model is usually specified with a "realistic" data set, then is used to demonstrate the implications of alternative input parameter and constraint specifications.

Solution of systems of equations: The final numerical use is as a technical device in empirical problems. MP can be used to develop such things as solutions to large systems of equations, equation fitting such that the estimated parameters minimize absolute deviations, or exhibit in all positive or all negative error terms. In this case, the ability of modern day solvers, i.e., computers, to treat problems with thousands of variables and constraints may be called to use. For example, a large USDA econometric model was solved for a time using a MP solver.

1.2.3. Algorithmic Development

Much of the mathematical programming related effort involves solution algorithm development. Formally, this is not a usage, but an enormous amount of work is done here as is evidenced by the many textbooks treating this topic. In such a setting the mathematical programming model is used as a vehicle for solution technique development. Work is also done on new formulation techniques and their ability to appropriately capture applied problems.

1.3. Book Plan

Mathematical programming in application consists, to a large degree, of applied linear programming. This book will not neglect that. Chapters 2, 3 and 4 will cover linear solution procedures, modeling, and sensitivity analysis. Chapter 5 will discuss integer programming and chapter 6 will cover nonlinear programming, then price endogenous programming in chapter 7 and portfolio theory in chapter 8 are followed.

1.4. Summary

- Mathematical programming (MP) refers to a set of procedures dealing with the analysis of optimization problems and it is thought of as a technique which decision makers can use to develop optimal values of the decision variables.

- Numerical usages of MP models fall into four subclasses: a) prescription of solutions, b) prediction of consequences, c) demonstration of sensitivity, and d) solution of systems of equations.

2. LINEAR PROGRAMMING

Key points:

Linear programming (LP): decision variables are chosen so that a linear function of the decision variables is optimized and a simultaneous set of linear constraints involving the decision variables is satisfied in the LP.

The basic LP formulation:

$$\max \quad z = \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for all } i = 1, \cdots, m$$
$$x_j \geq 0$$

LP assumptions are a) objective function appropriateness, b) decision variables appropriateness, c) constraint appropriateness, d) proportionality, e) additivity, f) divisibility, and g) certainty

2.1. Introduction

The most fundamental optimization problem (treated in this book) is the linear programming (LP) problem. LP was developed as a discipline in the 1940's, motivated initially by the need to solve complex planning problems, for example logistics, in wartime operations. Its development accelerated rapidly in the postwar period as many industries found valuable uses for LP. The founders of the subject are generally regarded as George B. Dantzig, who devised the simplex method in 1947, and John von Neumann, who established the theory of duality that same year. The Nobel Prize in Economics was awarded in 1975 to the mathematician Leonid Kantorovich (USSR) and the economist Tjalling Koopmans (USA) for their contributions to the theory of optimal allocation of resources, in which LP played a key role. Many industries use LP as a standard tool, e.g. to allocate a finite set of resources in an optimal way. Examples of important application areas include airline crew scheduling, shipping or telecommunication networks, oil refining and blending, and stock and bond portfolio selection. For the story about how LP began, refer to Dantzig (2002) "Linear Programming." *Operation Research* 50(1): 42-47.

In the LP problem, decision variables are chosen so that a linear function of the decision variables is optimized and a simultaneous set of linear constraints involving the decision variables is satisfied.

2.2. The Basic LP Problem

An LP problem contains several essential elements. First, there are decision variables, x_j, which

denotes the amount undertaken of the respective unknowns of which there are n ($j = 1, 2, \cdots, n$). Second is the linear objective function where the total objective value, z, equals $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$. Here c_j is the contribution (or profit margin) of each unit of x_j to the objective function. The problem is also subject to m constraints. An algebraic expression for the ith constraint is $a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i$ ($i = 1, 2, \cdots, m$), where b_i denotes the upper limit or right hand side imposed by the constraint and a_{ij} (technical coefficient) is the use of the items in the ith constraint by one unit of x_j. The c_j, b_i, and a_{ij} are the data (exogenous parameters) of the LP model.

Given these definitions, the LP problem is to choose $x_1, x_2, \cdots, x_n$ so as to

$$
\begin{array}{rccccccc}
\max \quad z = & c_1 x_1 & + & c_2 x_2 & + & \ldots & + & c_n x_n \\
\text{s.t.} & a_{11} x_1 & + & a_{12} x_2 & + & \ldots & + & a_{1n} x_n & \leq & b_1 \\
& a_{21} x_1 & + & a_{22} x_2 & + & \ldots & + & a_{2n} x_n & \leq & b_2 \\
& \vdots & & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
& a_{m1} x_1 & + & a_{m2} x_2 & + & \ldots & + & a_{mn} x_n & \leq & b_m \\
& x_1 & , & x_2 & , & \ldots & , & x_n & \geq & 0
\end{array}
$$

(2-1)

This formulation may also be expressed using the summation operators:

$$
\max \quad z = \sum_{j=1}^{n} c_j x_j
$$

(2-2) s.t.

$$
\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for all } i = 1, \cdots, m
$$

$$
x_j \geq 0
$$

Many variants have been posed of the above problem and applications span a wide variety of settings. For example, the basic problem could involve setting up a) a livestock diet determining how much of each feed stuff to buy so that total diet cost is minimized subject to constraints on minimum and maximum levels of nutrients b) a production plan where the firm chooses the profit maximizing level of production subject to resource (labor and raw materials) constraints or c) a minimum cost transportation plan determining the amount of goods to transport across each available route subject to constraints on supply availability and demand.

2.3. Basic LP Example – Joe's Van

For further exposition of the LP problem it is convenient to use an example. Consider the decision problem of Joe's van conversion shop. Suppose Joe takes plain vans and converts them into custom vans and can produce either fine or fancy vans. The decision modeled is how many of each van type to convert this week. The number converted this week by van type constitutes the decision variables.

We denote these variables as x_{fine} and x_{fancy}. Both types require a $25,000 plain van. Fancy vans sell for $37,000 and Joe uses $10,000 in parts to customize them yielding a profit margin of $2,000. Fine vans use $6,000 in parts and sell for $32,700 yielding profits of $1,700. Joe figures the shop can work on no more than 12 vans in a week. Joe hires 7 people including himself and operates 8 hours per day, 5 days a week and thus has at most 280 hours of labor available in a week. Joe also estimates that a fancy van will take 25 hours of labor, while a fine van will take 20 hours.

In order to set up Joe's problem as an LP, we must mathematically express the objective and constraint functions. Since the estimated profit (or profit margin) per fancy van is $2000 per van, then $2000x_{fancy}$ is the profit from all the fancy vans produced. Similarly, $1700x_{fine}$ is the profit from all the fine vans produced. The total profit from all van conversions is $z = 2000x_{fancy} + 1700x_{fine}$, where z is the total profit. This equation mathematically describes the total profit consequences of Joe's choice of the decision variables. Given that Joe wishes to maximize total profit, his objective is to determine the levels of the decision variables that

$$(2\text{-}3) \quad \max \quad z = 2000x_{fancy} + 1700x_{fine}$$

This is the underlined objective function of the LP model.

Joe's factory has limited amounts of capacity (size of the workshop) and labor. In this case, capacity and labor are resources which limit the allowable (or feasible) values of the decision variables. Since the decision variables are defined in terms of vans converted in a week, the total number of vans converted is $x_{fancy} + x_{fine}$. This sum must be less than or equal to the capacity available, 12 units. Similarly, total labor use is given by $25x_{fancy} + 20x_{fine}$ which must be less than or equal to the labor available, 280 hours. These two limits are called underlined constraints.

Finally, it makes no sense to convert a negative number of vans of either type; thus, x_{fancy} and x_{fine} are restricted to be greater than or equal to zero (non-negativity). Putting it all together, the LP model of Joe's problem is to choose the values of x_{fancy} and x_{fine} so as to:

$$
\begin{array}{rrrrrrl}
\max \quad z = & 2000x_{fancy} & + & 1700x_{fine} & & & \\
\text{s.t.} & x_{fancy} & + & x_{fine} & \leq & 12 & \text{[capacity constraint]} \\
& 25x_{fancy} & + & 20x_{fine} & \leq & 280 & \text{[labor constraint]} \\
& x_{fancy} & , & x_{fine} & \geq & 0 & \text{[non-negativity]}
\end{array}
$$

(2-4)

This is a underlined formulation of Joe's LP problem depicting the decision to be made (i.e. the choice of x_{fancy} and x_{fine}). The formulation also identifies the rules, commonly called underlined constraints, by which the decision is made and the objective which is pursued in setting the decision variables.

2.4. Additional LP Examples

2.4.1. Example 1

Suppose that a company makes two products (say, P and Q) using two machines (say, A and B). Each unit of P that is produced requires 50 minutes processing time on machine A and 30 minutes processing time on machine B. Each unit of Q requires 24 minutes processing time on machine A and 33 minutes processing time on machine B. Machine A will be available for 40 hours (2,400 minutes) and machine B is available for 35 hours (2,100 minutes). The profit margin of P is $25 and the profit margin of Q is $30. Company policy is to determine the production quantity of each product is such a way to maximize the total profit give that the available resources should not be exceeded.

Step 1: Defining Decision Variables
We often start with identifying decision variables. There are two decision variables, x_P, the number of units of P and x_Q, the number of units of Q.

Step 2: Choosing Objective Function
The company wants to maximize the profit. The profit per each unit of product P is $25 and Q is $30. Therefore, the total profit is $z = 25x_P + 30x_Q$.

Step 3: Identifying Constraints
The amount of time that machines A and B are available restrict the quantities to be produced. If we produce x_P units and x_Q units, machine A should be used for $50x_P + 24x_Q$ minutes since each unit of P requires 50 minutes processing time on machine A and each unit of Q requires 24 minutes processing time on machine A. Machine A is available for 40 hours. This impose the constraint, $50x_P + 24x_Q \leq 2400$. Similarly, the amount of time that machine B is available imposes the constraint, $30x_P + 33x_Q \leq 2100$.

Step 4: LP for the Example
All together with non-negativity,

$$
\begin{array}{rrcrcrl}
\max \quad z = & 25x_P & + & 30x_Q & & & \\
\text{s.t.} & 50x_P & + & 24x_Q & \leq & 2400 & \text{[machine A time]} \\
& 30x_P & + & 33x_Q & \leq & 2100 & \text{[machine B time]} \\
& x_P & , & x_Q & \geq & 0 & \text{[non-negativity]}
\end{array}
$$

(2-5)

2.4.2. Example 2 *(taken from McGuigan, Moyer and Harris, 1999)*

Consider the White Electronic a manufacturer of gas and electric clothes dryers. The problem is to determine the optimal level of output, x_{gas} and x_{ele} where x_{gas} is the number of gas clothes dryers produced and x_{ele} is the number of electric clothes dryers. Information about the production of these two products is summarized in Table 2-1.

Table 2-1: Resource and Profit Data for White Electronic

	Quantity of resources required per output		Quantity of resources available
	Gas dryer	Electric dryer	
Raw material (units)	20	40	400
Machine-processing time (hours)	5	2	40
Capacity of assembly division 1 (units)	1	0	6
Capacity of assembly division 2 (units)	0	1	9
Profit margins ($/unit)	100	60	

Production consists of a machining process that takes raw materials and converts them into unassembled parts. These are then sent to one of two divisions for assembly into the final product, Division 1 for gas dryer and Division 2 for electric dryer. As listed in Table 2-1, gas dryer requires 20 units of raw materials and 5 hours of machine-processing time, whereas electric dryer requires 40 units of raw material and 2 hours of machine-processing time. During the period, 400 units of raw material and 40 hours of machine-processing time are available. The capacities of the two assembly divisions during the period are 6 and 9, respectively.

Profit margins per unit is $100 per unit of gas dryer and $60 per unit of electric dryer. The profit margin represents the difference between the selling price and the variable cost per unit.

Step 1: Defining Decision Variables
There are two decision variables, x_{gas}, the number of units of gas dryers and x_{ele}, the number of units of electric dryers.

Step 2: Choosing Objective Function
The company wants to maximize the profit. The profit margin per each unit of gad dryer $100 and electric dryer is $60. Therefore, the total profit is $z = 100x_{gas} + 60x_{ele}$.

Step 3: Identifying Constraints
Consider first the raw material constraint. Production of x_{gas} units of gas dryers requires $20x_{gas}$ units of raw materials and production of x_{ele} units of electric dryers requires $40x_{ele}$ units of the same raw materials. The sum of these two quantities of raw materials must be less than or equal to the quantity available, which is 400 units; $20x_{gas} + 40x_{ele} \leq 400$.

The machine-processing time constraint can be developed in a like manner. Gas dryer requires $5x_{gas}$ hours and electric dryer requires $2x_{ele}$ hours. With 40 hours of processing time available, the following constraint is obtained; $5x_{gas} + 2x_{ele} \leq 40$.

The capacity of the two assembly divisions also limit output; for gas dryer, $x_{gas} \leq 6$ and for electric dryer, $x_{ele} \leq 9$.

Step 4: LP for the Example
All together with non-negativity,

$$
\begin{array}{rllllll}
\max & z = & 100x_{gas} & + & 60x_{ele} & & & \\
\text{s.t.} & & 20x_{gas} & + & 40x_{ele} & \leq & 400 & \text{[raw material]} \\
& & 5x_{gas} & + & 2x_{ele} & \leq & 40 & \text{[machine-processing]} \\
& & x_{gas} & & & \leq & 6 & \text{[division 1]} \\
& & & & x_{ele} & \leq & 9 & \text{[division 2]} \\
& & x_{gas} & , & x_{ele} & \geq & 0 & \text{[non-negativity]}
\end{array}
$$

(2-6)

2.5. Other Forms of the LP Problem

Not all LP problems will naturally correspond to the above form. Other representations of LP models are:

- Objectives which involves minimize instead of maximize i.e., $\min \quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

- Constraints which are "greater than or equal to" instead of "less than or equal to"; i.e., $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i$

- Constraints which are strict equalities; i.e., $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$

- Variables without non-negativity restriction i.e., x_j can be unrestricted in sign,

- Variables required to be non-positive i.e., $x_j \leq 0$.

2.5.1. *Cost Minimization* (taken from McGuigan, Moyer and Harris, 1999)

Suppose that the Silverado Mining Company owns two different mines (say mines A and B) for producing uranium ore. The two mines are located in different areas and produce different qualities of uranium ore. After the ore is mined, it is separated into three grades – high-, medium-, and low-grade. Information concerning the operation of the two mines is shown in Table 2-2. Mine A produces 0.75 tons of high-grade ore, 0.25 tons of medium-grade ore, and 0.50 tons of low-grade ore per hour. Likewise, Mine B produces 0.25, 0.25 and 1.50 tons of high-, medium-, and low-grade ore per hour, respectively. The firm has contracts with uranium-processing plants to supply a minimum of 36 tons of high-grade ore, 24 tons of medium-grade ore, and 72 tons of low-grade ore per week. Finally as shown in the bottom row of Table 2-2, it costs the company $50 per hour to operate Mine A and $40 per hour to operate Mine B. The company wishes to determine the number of hours per week it should operate each mine to minimize the total cost of fulfilling its supply contracts.

Step 1: Defining Decision Variables
There are two decision variables, x_A, the number of hours per week that Mine A is operated and x_B, the number of hours per week that Mine B is operated.

Table 2-2: Resource and Profit Data for Silverado Mining

	Outputs (tons per hour) Mine		Requirements (tons per week)
Type of ore	**A**	**B**	
High-grade	0.75	0.25	36
Medium-grade	0.25	0.25	24
Low-grade	0.50	1.50	72
Operating cost ($/hour)	50	40	

Step 2: Choosing Objective Function

The company wants to minimize the total cost per week from the operation of the two mines and the total cost is the sum of the operating cost per hour of each mine times the number hours per week that each mine is operated. The total cost is $z = 50x_A + 40x_B$.

Step 3: Identifying Constraints

The company's contracts with uranium-processing plants require it to operate the two mines for a sufficient number of hours to produce the required amount of each grade of uranium ore such that $0.75x_A + 0.25x_B \geq 36$ (high), $0.75x_A + 0.25x_B \geq 36$ (medium), and $0.75x_A + 0.25x_B \geq 36$ (low).

Step 4: LP for the Example

All together with non-negativity,

$$
\begin{array}{rlrlcll}
\min \quad z = & 50x_A & + & 40x_B & & & \\
\text{s.t.} \quad & 0.75x_A & + & 0.25x_B & \geq & 36 & \text{[high-grade ore]} \\
(2\text{-}7) \quad & 0.25x_A & + & 0.25x_B & \geq & 24 & \text{[medium-grade ore]} \\
& 0.50x_A & + & 1.50x_B & \geq & 72 & \text{[low-grade ore]} \\
& x_A & , & x_B & \geq & 0 & \text{[non-negativity]}
\end{array}
$$

2.6. Assumptions of LP

LP problems embody seven important assumptions relative to the problem being modeled. The first three involve the appropriateness of the formulation; the last four the mathematical relationships within the model.

2.6.1. Objective Function Appropriateness

This assumption means that the objective function is the sole criteria for choosing among the feasible values of the decision variables. Satisfaction of this assumption can often be difficult as, for example, Joe might base his van conversion plan not only on profit but also on risk exposure, availability of vacation time, etc. The <u>multi-objective</u> chapter covers the relaxation of this assumption.

2.6.2. Decision Variable Appropriateness

A key assumption is that the specification of the decision variables is appropriate. This assumption requires that

- The decision variables are all fully manipulatable within the feasible region and are under the control of the decision maker.
- All appropriate decision variables have been included in the model.

2.6.3. Constraint Appropriateness

The third appropriateness assumption involves the constraints. Again, this is best expressed by identifying sub-assumptions:

- The constraints fully identify the bounds placed on the decision variables by resource availability, technology, the external environment, etc. Thus, any choice of the decision variables, which simultaneously satisfies all the constraints, is admissible.
- The resources used and/or supplied within any single constraint are homogeneous items that can be used or supplied by any decision variable appearing in that constraint.
- Constraints have not been imposed which improperly eliminate admissible values of the decision variables.
- The constraints are inviolate. No considerations involving model variables other than those included in the model can lead to the relaxation of the constraints.

Relaxations and/or the implications of violating these assumptions are discussed throughout the text.

2.6.4. Proportionality

Variables in LP models are assumed to exhibit proportionality. Proportionality deals with the contribution per unit of each decision variable to the objective function. This contribution is assumed constant and independent of the variable level. Similarly, the use of each resource per unit of each decision variable is assumed constant and independent of variable level. There are no economies of scale. For example, in the general LP problem, the net return per unit of x_j produced is c_j. If the solution uses one unit of x_j, then c_j units of return are earned, and if 100 units are produced, then returns are $100c_j$. Under this assumption, the total contribution of x_j to the objective function is always proportional to its level.

This assumption also applies to resource usage within the constraints. Joe's labor requirement for fine vans was 25 hours/van. If Joe converts one fine van he uses 25 hours of labor. If he converts 10 fine vans he uses 250 hours (25*10). Total labor use from van conversion is always strictly proportional to the level of vans produced.

Economists encounter several types of problems in which the proportionality assumption is grossly violated. In some contexts, product price depends upon the level of production. Thus, the contribution per unit of an activity varies with the level of the activity. Methods to relax the proportionality

assumption are discussed in the nonlinear approximations, price endogenous, and risk chapters. Another case occurs when fixed costs are to be modeled. Suppose there is a fixed cost associated with a variable having any non-zero value (i.e., a construction cost). In this case, total cost per unit of production is not constant. The integer programming chapter discusses relaxation of this assumption.

2.6.5. Additivity

Additivity deals with the relationships among the decision variables. Simply put their contributions to an equation must be additive. The total value of the objective function equals the sum of the contributions of each variable to the objective function. Similarly, total resource use is the sum of the resource use of each variable. This requirement rules out the possibility that interaction or multiplicative terms appear in the objective function or the constraints. For example, in Joe's van problem, the value of the objective function is 2,000 times the fancy vans converted plus 1,700 times the fine vans converted. Converting fancy vans does not alter the per van net margin of fine vans and vice versa. Similarly, total labor use is the sum of the hours of labor required to convert fancy vans and the hours of labor used to convert fine vans. Making a lot of one van does not alter the labor requirement for making the other.

In the general LP formulation, when considering variables x_j and x_k, the value of the objective function must always equal c_j times x_j plus c_k times x_k. Using x_j does not affect the per unit net return of x_k and vice versa. Similarly, total resource use of resource i is the sum of $a_{ij}x_j$ and $a_{ik}x_k$. Using x_j does not alter the resource requirement of x_k. The nonlinear programming and price endogenous chapters present methods of relaxing this assumption.

2.6.6. Divisibility

The problem formulation assumes that all decision variables can take on any non-negative value including fractional ones; (i.e., the decision variables are continuous). In the Joe's van shop example, this means that fractional vans can be converted; e.g., Joe could convert 11.2 fancy vans and 0.8 fine vans. This assumption is violated when non-integer values of certain decision variables make little sense. A decision variable may correspond to the purchase of a tractor or the construction of a building where it is clear that the variable must take on integer values. In this case, it is appropriate to use integer programming.

2.6.7. Certainty

The certainty assumption requires that the parameters c_j, b_i, and a_{ij} be known constants. The optimum solution derived is predicated on perfect knowledge of all the parameter values. Since all exogenous factors are assumed to be known and fixed, LP models are sometimes called non-stochastic as contrasted with models explicitly dealing with stochastic factors. This assumption gives rise to the term "deterministic" analysis. The exogenous parameters of a LP model are not usually known with certainty. In fact, they are usually estimated by statistical techniques. Thus, after developing a LP model, it is often useful to conduct sensitivity analysis by varying one of the exogenous parameters

and observing the sensitivity of the optimal solution to that variation. For example, in the van shop problem the net return per fancy van is $2,000, but this value depends upon the van cost, the cost of materials and the sale price all of which could be random variables.

Considerable research has been directed toward incorporating uncertainty into programming models. We devote a chapter to that topic.

3. SOLUTION OF LP PROBLEMS

Key points:

The simplex method is a general procedure for solving LP problems.

MS Excel Solver has the capability to solve LP problems.

The shadow price on a particular constraint represents the change in the value of the objective function per unit increase in the RHS value of the corresponding. Reduced cost is the shadow price associated with the non-negativity constraint.

The duality indicates that the maximum value of the primal profit function will also be equal to the minimum value of the dual imputed value function (cost of resources employed).

3.1. Introduction

Linear programming solution has been the subject of many articles and books. Complete coverage of LP solution approaches is beyond the scope of this book and is present in many other books. However, an understanding of the basic LP solution approach and the resulting properties are of fundamental importance. Thus, we cover LP solution principles from a graphical perspective demonstrating the simplex algorithm, which is a primary way to solve (large) LP problems.

3.2. Simplex Method

The simplex method is a general procedure for solving LP problems which is developed by G. Dantzig in 1947. The simplex method constructs corner-point feasible solutions (CPF solutions) and then investigates each CPF solution successively until optimum is reached. The simplex method has proved to be a remarkable efficient method to solve huge problems on computer. The simplex method is an algebraic procedure but its underlying concepts are geometric. We focus on a geometric viewpoint first.

Recall Joe's van example in chapter 2:

$$
\begin{array}{rrrrrrll}
\max & z = & 2000x_{\text{fancy}} & + & 1700x_{\text{fine}} & & & \\
\text{s.t.} & & x_{\text{fancy}} & + & x_{\text{fine}} & \leq & 12 & \text{[capacity constraint]} \\
& & 25x_{\text{fancy}} & + & 20x_{\text{fine}} & \leq & 280 & \text{[labor constraint]} \\
& & x_{\text{fancy}} & , & x_{\text{fine}} & \geq & 0 & \text{[non-negativity]}
\end{array}
$$

(3-1)

3.2.1. Graphical Representation of the Decision Space

We now look at the constraints of the above LP problem. Note from equation (3-1) that each of the decision variable must be greater than or equal to zero. Therefore we need only graph the upper right-hand (first quadrant). Figure 3.1 illustrates the capacity constraint as given by the first constraint in equation (3-1).
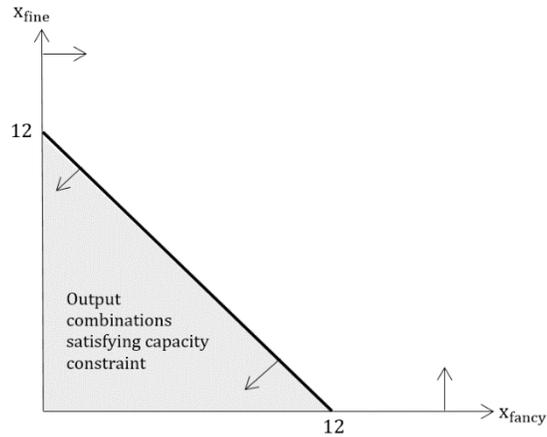
Figure 3-1. Graphical Representation of Capacity Constraint

The maximum quantity of capacity that may be used occurs when the inequality is satisfied as an equality; in other words, the set of points that satisfy the equation; $x_{fancy} + x_{fine} = 12$, or

(3-2) $x_{fine} = 12 - x_{fancy}$

Because it is possible to use less than the amount of capacity available, any combination of outputs lying on or below (directed by arrows in Figure 3-1) the line (shaded area) will satisfy the capacity constraint. Similarly, the second constraint on the labor, $25x_{fancy} + 20x_{fine} \leq 280$, yields the combinations of x_{fancy} and x_{fine} that lie on or below the line, $x_{fine} = \frac{280}{20} - \frac{25}{20}x_{fancy}$, shown in Figure 3-2 (shaded area).
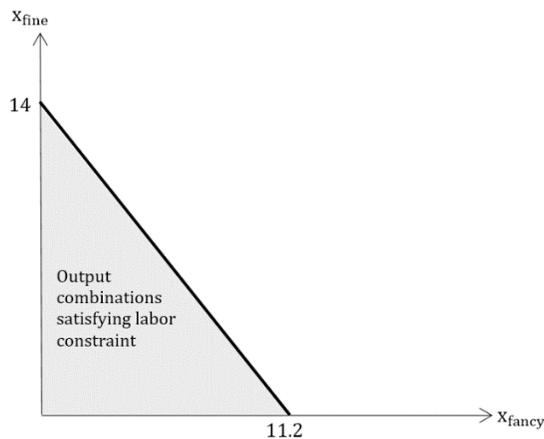
Figure 3-2. Graphical Representation of Labor Constraint
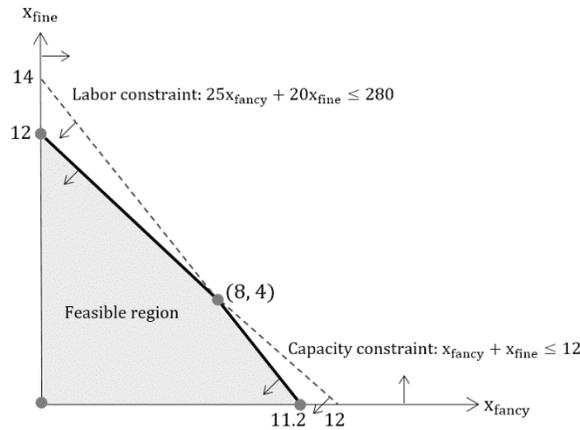
Figure 3-3. Graphical Representation of the Feasible Region

Combining all the constraints yields the feasible solution space (feasible region) to the problem, where all of the values of the decision variables, x_{fancy} and x_{fine}, simultaneously satisfy the constraints. It can be represented geometrically by the shaded area given in Figure 3-3. The arrows associated with each line show the direction indicated by the inequality sign in each constraint. Four dots, (0, 0), (11.2, 0), (8, 4) and (0, 12), are corner-point feasible (CPF) solutions. Any production alternative (combination of x_{fancy} and x_{fine}) not within the feasible region must violate at least one of the constraints of the problem. Among these feasible production alternatives, we want to find the values of the decision variables x_{fancy} and x_{fine} that maximize Joe's profit, $z = 2000x_{fancy} + 1700x_{fine}$.

3.2.2. Finding an Optimal Solution

To find an optimal solution, we first note that any point in the interior of the feasible region cannot be an optimal solution since the contribution can be increased by increasing either x_{fancy} or x_{fine} or both. To make this point more clearly, let's rewrite the objective function in terms of x_{fine} as follows:

$$(3\text{-}3) \quad x_{fine} = \frac{z}{1700} - \frac{2000}{1700}x_{fancy}$$

If z is held fixed at a given constant value, this expression represents a straight line, where $\frac{z}{1700}$ is the intercept with the x_{fine} axis (i.e., the value of x_{fine} when $x_{fancy} = 0$), and $-\frac{2000}{1700} \approx -1.1765$ is the slope (i.e., the change in the value of x_{fine} corresponding to a unit increase in the value of x_{fancy}). Note that the slope of this straight line is constant, independent of the value of z. As the value of z increases, the resulting straight lines move parallel to themselves in a northeasterly direction away from the origin (because the intercept $\frac{z}{1700}$ increases when z increases, and the slope is constant at $-\frac{2000}{1700}$).
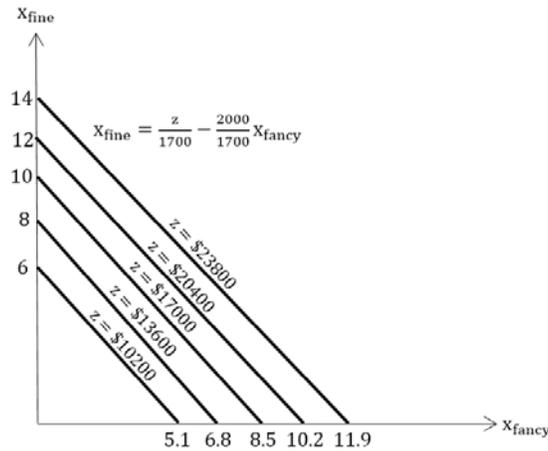
Figure 3-4. Joe's Van Iso-profit Lines

Figure 3-4 shows some of these parallel lines for specific values of z, for example, z = $10,200. Each of the lines shown in Figure 3-4 is an iso-profit line, meaning that each combination of output level, (x_{fancy}, x_{fine}) lying on a given line has the <u>same</u> total profit. For example, z = $17,000 iso-profit line includes such output combinations as points between (0, 10) and (8.5, 0). The goal of profit maximization can be interpreted graphically to find an output combination that falls on as high as iso-profit lines as possible.

Combining Figures 3-3 and 3-4 yields the output combination point within the feasible region that lies on the highest possible iso-profit line. At the point labeled E, the line intercepts the farthest point from the origin within the feasible region, and the contribution z cannot be increased any more. Therefore, point E represents the optimal solution (Figure 3-5).
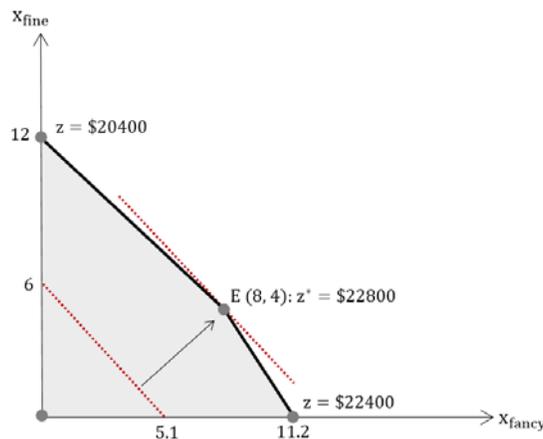
Figure 3-5. Optimal Solution

Since reading the graph may be difficult, we can compute the values of the decision variables by recognizing that point E is determined by the intersection of the capacity constraint and the labor constraint. Solving these constraints,

$$(3\text{-}4) \quad \begin{array}{rcrcll} x_{fancy} & + & x_{fine} & = & 12 & \text{[capacity constraint]} \\ 25x_{fancy} & + & 20x_{fine} & = & 280 & \text{[labor constraint]} \end{array}$$

yields $x_{fancy} = 8$, $x_{fine} = 4$ and substituting these values to the objective function yields $z = \$22{,}800$ as the maximum contribution that can be attained.

Note that the optimal solution is at a corner point, or vertex, of the feasible region. This turns out to be a general property of LP: if a problem has an optimal solution, there is always a vertex, <u>one of CPF solution that is optimal</u>. The simplex method for finding an optimal solution to a general LP models exploits this property by starting at a vertex and moving from vertex to vertex, improving the value of the objective function with each move. In Figure 3.5, the values of the decision variables and the associated value of the objective function are given for each vertex of the feasible region. Any procedure that starts at one of the vertices and looks for an improvement among adjacent vertices would also result in the solution labeled E. An optimal solution of a LP in its simplest form gives the value of the criterion function, the levels of the decision variables, and the amount of slack or surplus in the constraints. In Joe's van example, the criterion was maximum profit, which turned out to be $z^* = \$22{,}800$; the level of the decision variables are $x^*_{fancy} = 8$ and $x^*_{fine} = 4$.

3.2.3. Joe's Van and Simplex

Each constraint boundary is a line that forms the boundary of what is permitted by the corresponding constraint. As said, <u>corner-point feasible solutions</u> (CPF solutions) are corner-point solutions within feasible region; e.g., (0, 0), (0, 12), (8, 4), and (11.2, 0) (Figure 3-5). Again, when an optimal solution of a LP exists, it occurs one of CPF solutions. When two CPF solutions are adjacent to each other when they are connected by a line segment, for example, (0, 0) is adjacent to (0, 12) and (11.2, 0); (8, 4) is adjacent to (0, 12) and (11.2, 0) as well.

Simplex method begins with any CPF solution, usually origin, move to an adjacent CPF solution with a better objective value, and continue until no adjacent CPF solution has a better objective value. If a CPF solution has no adjacent CPF solutions that are better (as measured by z), then it must be an optimal solution.

Solving the example:

- Initialization: Choose (0, 0) as the initial CPF solution to examine (this is a convenient choice) and measure z, z = 0
    - Optimality test: (0, 0) is not optimal because adjacent CPF solutions are better; (0, 12) gives z = 20,400 and (11.2, 0) has z = 22,400.
- Iteration 1: Move to a better adjacent CPF solution; in this case, move to (11.2, 0)

- o Optimality test: (11.2, 0) is not an optimal solution because adjacent CPF solution, (8,4) is better; (8, 4) gives z = 22,800
- Iteration 2: Move to a better adjacent CPF solution, (8, 4)
  - o Optimality test: (8, 4) is an optimal solution because no adjacent CPF solution is better.

3.2.4. Solution for White Electronic

Recall the additional LP example in equations (2-6) in Chapter 2, White Electronic dryer example:

$$
\begin{aligned}
\max \quad z = \quad & 100x_{gas} \; + \; 60x_{ele} \\
\text{s.t.} \quad & 20x_{gas} \; + \; 40x_{ele} \; \leq \; 400 \quad \text{[raw material]} \\
& 5x_{gas} \; + \; 2x_{ele} \; \leq \; 40 \quad \text{[machine-processing]} \\
& x_{gas} \qquad\qquad\quad \leq \quad 6 \quad \text{[division 1]} \\
& \qquad\qquad\quad x_{ele} \; \leq \quad 9 \quad \text{[division 2]} \\
& x_{gas} \quad , \quad x_{ele} \; \geq \quad 0 \quad \text{[non-negativity]}
\end{aligned}
$$

(3-5)

The feasible area and CPF solutions are identified following the discussion above. Figure 3-6 presents the graphical solution of the White Electronic dryer example in equation (3-5). The optimal solution is $x_{gas}^* = 5$, $x_{ele}^* = 7.5$ and $z^* = 950$.
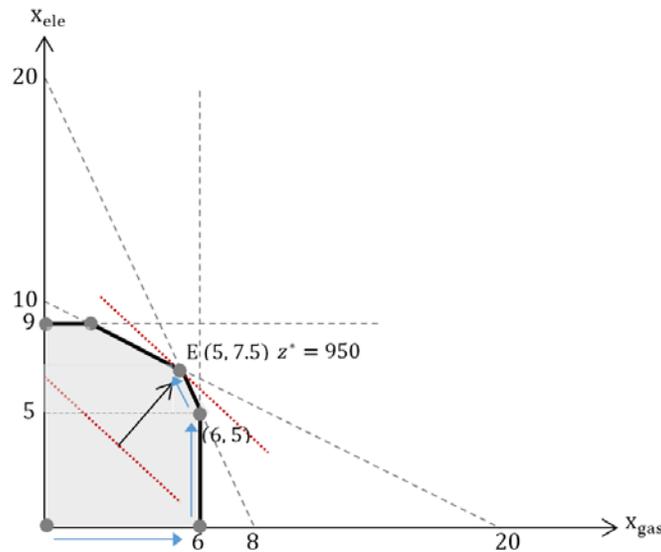
Figure 3-6. Finding Optimal Solution

20

Simplex method solves the problem as follows:

- Initialization: Choose (0, 0) as the initial CPF solution to examine and measure z, $z = 0$
  - Optimality test: (0, 0) is not optimal because adjacent CPF solutions are better; (0, 9) gives z = 540 and (6, 0) has z = 600.

- Iteration 1: Move to a better adjacent CPF solution; in this case, move to (6, 0)
  - Optimality test: (6, 0) is not an optimal solution because adjacent CPF solution, (6, 5) is better, which gives z = 900

- Iteration 2: Move to a better adjacent CPF solution, (6, 5)
  - Optimality test: (6, 5) is not an optimal solution because adjacent CPF solution, (5, 7.5) is better, which gives z = 950

- Iteration 3: Move to a better adjacent CPF solution, (5, 7.5)
  - Optimality test: (5, 7.5) is an optimal solution because no adjacent CPF solution, for example adjacent CPF (2, 9) has z = 740.

3.2.5. Solution for Silverado Mining

How about the minimization? Recall the minimization example in equation (2-7) Chapter 2, Silverado Mining Company.

$$
\begin{aligned}
\min \quad z = \quad & 50x_A \; + \quad 40x_B \\
\text{s.t.} \quad & 0.75x_A \; + \; 0.25x_B \; \geq \; 36 \quad \text{[high-grade ore]} \\
(3\text{-}6) \quad & 0.25x_A \; + \; 0.25x_B \; \geq \; 24 \quad \text{[medium-grade ore]} \\
& 0.50x_A \; + \; 1.50x_B \; \geq \; 72 \quad \text{[low-grade ore]} \\
& \quad x_A \quad , \qquad x_B \; \geq \quad 0 \quad \text{[non-negativity]}
\end{aligned}
$$

The feasible area and CPF solutions are identified (Figure 3-7). Note that constraints in the problem are greater than equal to and thus the feasible region is the above the constraints lines. The optimal solution occurs one of CPF solutions and thus evaluate $(x_A, x_B) = (0, 144), (24, 72), (72, 24)$, and $(144, 0)$ in terms of the objective function and pick the minimum one. The optimal solution is $(x_A^*, x_B^*) = (24, 72)$ and $z^* = 4080$. To determine the optimal solution using the iso-cost line,

$$
(3\text{-}7) \quad x_B = \frac{z}{40} - \frac{50}{40}x_A
$$

If z is held fixed at a given constant value, this expression represents a red dotted straight line in Figure 3-7, where $\frac{z}{40}$ is the intercept with the x_B axis and $-\frac{50}{40} = -1.25$ is the slope. As the value of z decreases (remember it is minimization), the iso-cost line moves toward the origin and it touches the pointe labeled E, which is the optimal solution.
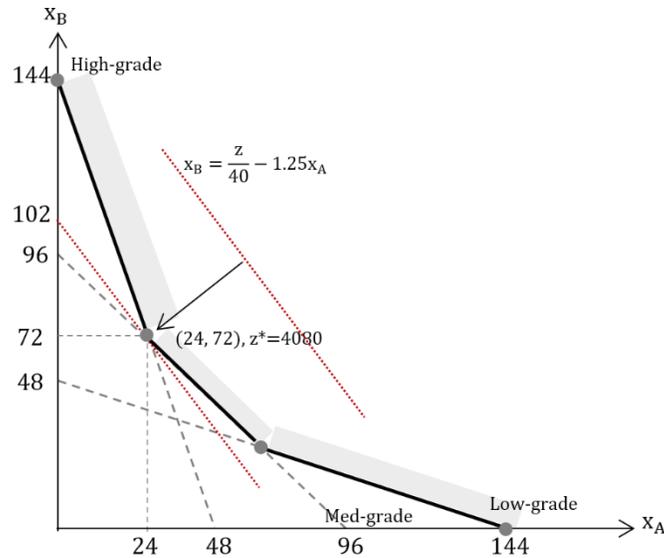
Figure 3-7. Finding Optimal Solution

The simplex method is not applicable for the minimization; equation (3-6) is converted to the maximization problem using duality and, in turn, we can use the simplex method. For the duality, see section 3.5.

3.3. LP on Excel

Microsoft Excel Solver (Add-in) has the capability to solve LP problems and it can be used to solve problems with up to 200 decision variables. The Solver Add-in is a Microsoft Office Excel add-in program that is available when you install Microsoft Office or Excel. To use the Solver Add-in, however, you first need to load it in Excel. The process is slightly different for Mac or PC users.

3.3.1. Adding Solver Add-in

Windows PC: load Excel with a new Excel workbook and click "File" tab. Find "Options" Excel Options widow appears. Click Add-ins, and then in the Manage box, select Excel Add-ins and click Go. In the Add-ins available box, select the Solver Add-in check box, and then click OK. After you load Solver, the Solver command is available in the "Data" tab.

File > Options > Add-ins > Manage: Excel Add-ins, Go > Solver Add-in > OK

Mac PC: execute Excel with a new Excel workbook and click "Tools" menu and select "Excel Add-ins" Add-ins available box appears and check Solver Add-in. Click OK. After you load Solver, the Solver command is available in the "Data" tab.

Tools > Excel Add-ins > Solver Add-in > OK

3.3.2. Formulating LP on Excel Spreadsheet

Recall Joe's van example in equation (3-1). To begin, we enter heading for each decision variable, x_{fancy} and x_{fine} in the rage B3:C3 (Panel A in Figure 3-4). In the rage B4:C4, we input *trial values* for the number of vans converted (any values will work).

In the range B5:C5, we enter the profit margins from each van type and then we compute the profit of the van conversion in the cell E5 with the formula

(3-8) $= B4 * B5 + C4 * C5$

But it is usually easier to enter the formula using the SUMPRODUCT function:

(3-9) $= \text{SUMPRODUCT}(B4\!:\!C4, B5\!:\!C5)$

The =SUMPRODUCT function requires two ranges as inputs. The first cell in the first range is multiplied by the first cell in the second range, then the second cell in the first range is multiplied by the second cell in the second range, and so on (Panel B in Figure 3.8).

Panel A

	A	B	C	D	E	F	G
1	Joe's Van LP						
2							
3		Fancy	Fine				
4		2	2				
5							
6							
7							
8							
9							
10							
11							
12							
13							

Panel B

	A	B	C	D	E	F	G
1	Joe's Van LP						
2							
3		Fancy	Fine				
4		2	2				
5		2000	1700		=SUMPRODUCT(B4:C4,B5:C5)		
6							
7							
8							
9							
10							
11							
12							
13							

Panel C

	A	B	C	D	E	F	G
1	Joe's Van LP						
2							
3		Fancy	Fine				
4		2	2				
5		2000	1700		7400		
6							
7		1	1				
8		25	20				
9							
10							
11							
12							
13							

Panel E

	A	B	C	D	E	F	G	H
1	Joe's Van LP							
2								
3		Fancy	Fine					
4		2	2					
5		2000	1700		7400			
6								
7		1	1		=SUMPRODUCT($B$4:$C$4,B7:C7)			
8		25	20		90	<=		
9								
10								
11								
12								
13								
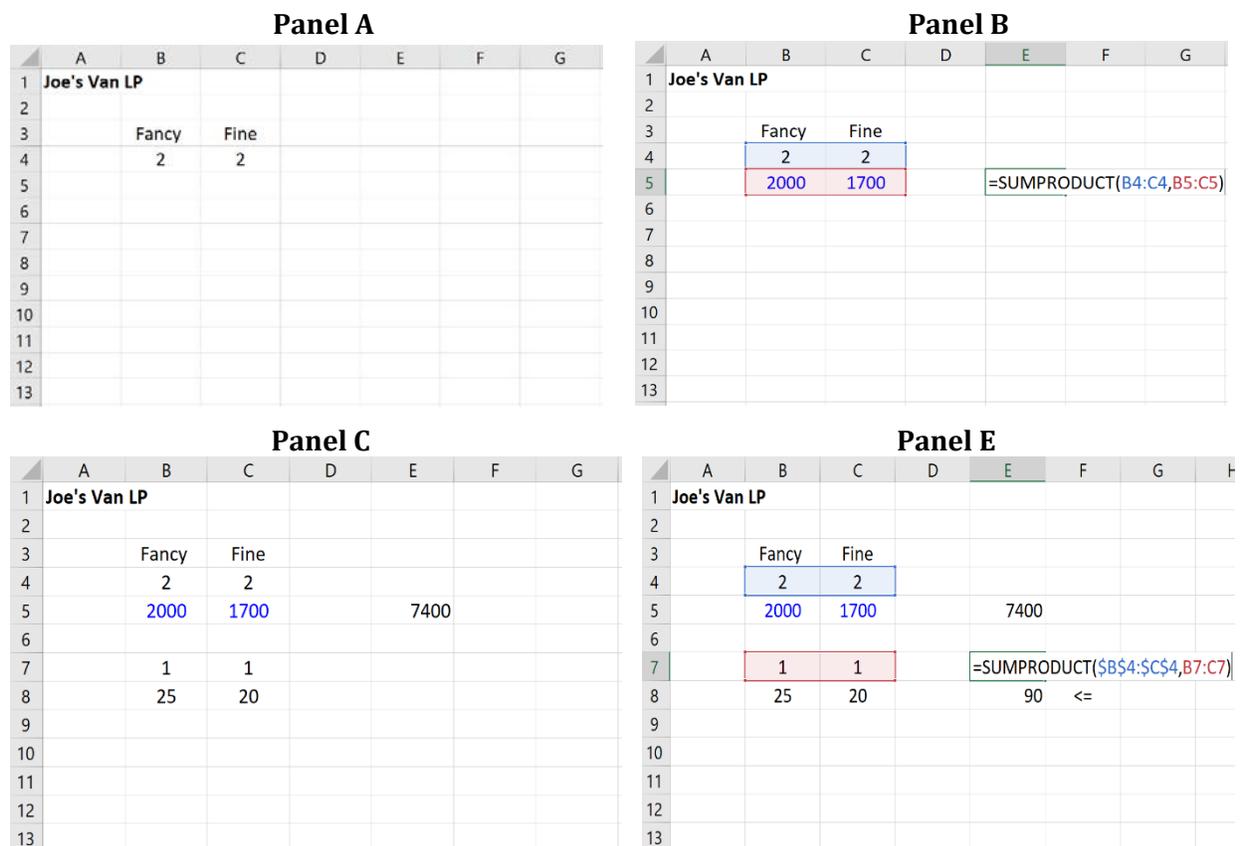
Figure 3-8. Formulating LP on Excel (1)

Figure 3-9. Formulating LP on Excel (2)

The next step required to set up our LP in Excel is to set up our constraints for capacity and labor. To begin, we recreate the table in Excel in the range B7:C8 that defines capacity and labor requirements to convert vans (Panel C in Figure 3-8). In the range E7:E8, enter the formula to calculate resource used with inequalities in column E (Panel D in Figure 3-8). Adding the capacity and labor available in the column G, formulating LP on excel is done. Cells in the range B4:C4 are "changing variable cells" which will be determined by Excel Solver. The cell E5 contains the objective function ("target cell").

Adding proper headings and some colors (green = changing cells, yellow = cells with formulas, for example), we have the Excel LP model for Joe's Van (Figure 3.9)

3.3.3. Solving LP model using the Solver

The Solver Parameters Dialog Box (Figure 3.10) is used to describe the optimization problem to Excel. To open the Dialog Box, select Data tab and Solver.

The way we set up the problem in Excel will make it easy for us to fill in each of the components of this Parameters Dialog Box so Solver identify the optimal solution. Firs we fill in the "Set Objective" box by clicking on the cell in the spreadsheet that calculates the objective function, cell E5. Next we use the radio buttons below to identify the type of problem we are solving, a MAX or MIN. Here we want to maximize the profit and select Max. We need to identify the decision variables (Changing Variable Cells). After clicking into the "By Changing Variable Cells" box, we can select the decision variable cells in the LP, B4:C4. This tells Solver that it can change the number of fancy vans and fine vans (Figure 3-10).
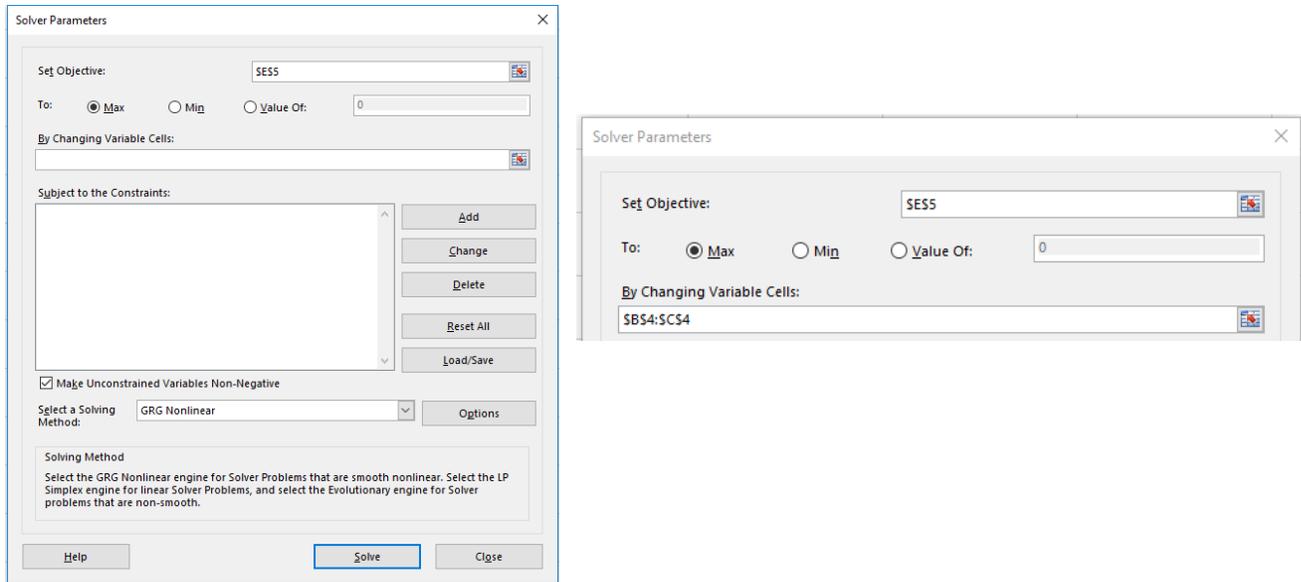
Figure 3-10. Solver Parameters Dialog Box

We need to add two constraints to Solver to ensure our solution does not violate any of them. On the right-hand side of the window, there is a button to "Add" a constraint. After clicking on this, a box will appear that allows us to add our constraints (Figure 3-11). We can use the "Cell Reference" box to input the range of cells with formulas (range E7:E8). There are several options for constraint type: <=, >=, =, int (integer), bin (binary), dif (all different). After adjusting the constraint type to be less than or equal to (<=) we can click on the cell referencing the resources available (RHS, range G7:G8).

The "Change" button in the Solver Parameter Dialog box allows you to modify a constraint already entered and "Delete" allows you to delete a previously entered constraint. If you need to add more constraints, choose "Add." After adding all constraints, the SOLVER Parameters Dialog Box looks like Figures 3-12.
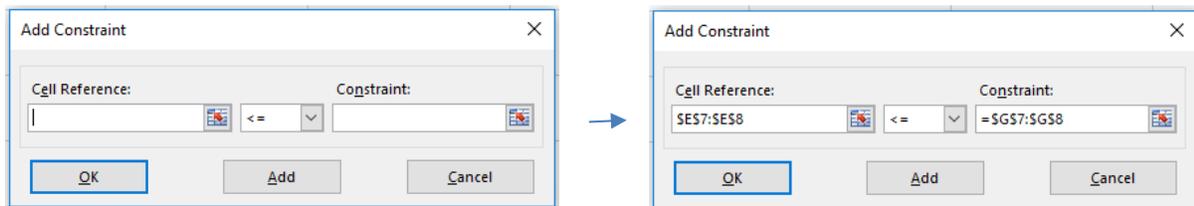
Figure 3-11. Add Constraint Dialog Box
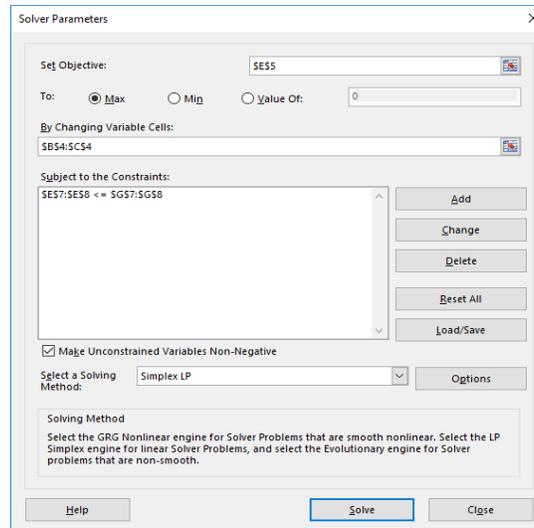
Figure 3-12. Final Solver Parameters Dialog Box

Note the checked box titled "Make Unconstrained Variables Non-Negative" that allows us to capture non-negativity constraints. Additionally, you should change the "Select a Solving Method" to Simplex LP when you are solving a linear program. The other options allow for solutions for nonlinear programs (will be discussed in later chapters).

Finally, click "Solve" button for the solution (Figure 3-12). After you solve, the Parameters Dialog Box will close and the decision variables will change to the optimal solution. Because we referenced these cells in all our calculations, the objective function and constraints will also change. When Excel finds an optimal solution, Solve Results box will appear in Figure 3-13. Solver Results will indicate that Solver found a solution and all constraints and optimality conditions are satisfied. "Keep Solver Solution" is checked by default. Click OK for now. We will re-visit the Solver Results box later.

Figure 3-13. Solver Results Box

Figure 3-14. Optimal Solution

Final Excel spreadsheet contains the optimal solution, value of the objective function, and resource used as in Figure 3-14.

It is desirable to add some more texts with colors and boxes as in Figure 3-15 for report writing after solving the model. We will explore these in detail in Chapter 4. Note that cells with green color are changing cells where Excel Solver displays the optimal solution and yellow cells contain all the formulas in the model and updated based on values in green cells (of course, coloring and adding texts are your choice).

Figure 3-15. Optimal Solution and Explanatory Texts

3.4. Solutions, Interpretation and Sensitivity Analysis

LP solutions are composed of a number of elements. In this chapter we discuss general solution interpretation, common solver solution format and contents, special solution cases and sensitivity analysis.

3.4.1. General Solution Interpretation

Recall the Joe's van example (Figure 3-15). Excel Solver provides optimal solution which maximizes Joe's profit: $x^*_{fancy} = 8$, $x^*_{fine} = 4$, and $z^* = \$22800$. At this level of production, Joe uses 12 units of capacity (capacity constraint is binding) and 280 units of labor (labor constraint is binding as well).

When Excel finds an optimal solution, Solve Results box will appear (Figure 3-13). There are three options in Reports box: Answer, Sensitivity, and Limits. An Answer Report will be generated after selecting Answer option in the Report box and clicking OK button. Figure 3-16 shows the Answer Report. It provides the basic information for the optimal solution in three output sections:

- Objective Cell (Max) section: the optimal value of the objective function, z, determined by Solver, 22800 in Final Value. Cell address, $E$5, is printed where the objective function is located. Note that Answer Report prints "Profit z" in Name column, which are located in Joe's van model, D5 and E4 (red texts in Figure 3-15).
- Variable Cells section: the optimal values of the two decision variables with cell addresses and names, Fancy = 8 and Fine = 4. "Contin" in integer column indicates that both variables are continuous. Excel picks up names from cell A4 and B3:C3 in the model (red texts in Figure 3-15).
- Constraints section: information relating to the constraints and the amount of resource consumed by the optimal solution. The information relating to constraints comes in the form of a constraint designation found in the Status column, as either Not Binding or Binding. When a constraint is binding Slack is zero. When a constraint is not binding slack is positive number.

In addition, upper left corner of the Answer Report also presents version of Excel, name of worksheet, date of report created, information about result, solver option (not shown in Figure 3-16).

	Cell	Name	Original Value	Final Value		
14	**Objective Cell (Max)**					
15	**Cell**	**Name**	**Original Value**	**Final Value**		
16	$E$5	Profit z	0.00	22800.00		
19	**Variable Cells**					
20	**Cell**	**Name**	**Original Value**	**Final Value**	**Integer**	
21	$B$4	Production of Fancy	0.00	8.00	Contin	
22	$C$4	Production of Fine	0.00	4.00	Contin	
25	**Constraints**					
26	**Cell**	**Name**	**Cell Value**	**Formula**	**Status**	**Slack**
27	$E$7	Capacity Resouce used	12.00	$E$7<=$G$7	Binding	0
28	$E$8	Labor Resouce used	280.00	$E$8<=$G$8	Binding	0

Figure 3-16. Answer Report

3.4.2. Sensitivity Analysis

Ranging analysis is the widely utilized tool for analyzing how much a LP model and its solution can be altered without changing the interpretation of the solution. Ranging analysis deals with the "what if" question and provides valuable insights of the model. Ranging analysis of right-hand-side (b_i) and objective function coefficients (c_j) are common; Excel Solver has options to conduct ranging analyses, which is the option Sensitivity in Reports box in Solver Results Box (Figure 3-9).

3.4.2.1. *Shadow Price and Reduced Cost*

Now consider the question of making one-at-a-time changes in the right-hand-side values of the constraints. Suppose that Joe has one more unit of capacity limit, 13 units instead of 12 units (one unit upward change). It is illustrated in Figure 3-17. In Figure 3-17, new capacity constraint, $x_{fancy} + x_{fine} \leq 13$ is added which expand the feasible region. The original optimal solution (8, 4) is not optimal simply because it is on the constraint, not the corner point feasible (CPF) solution. As shown in Figure 3-17, the new optimal solution is $x_{fancy} = 4$ and $x_{fine} = 9$ and the new maximized objective function value is $z_1^* = 23300$. The difference, $\Delta z = z_1^* - z_0^* = 23300 - 22800 = 500$. In other words, when capacity limit is increased upward from 12 to 13, z would increase by $500 with 4 less fancy vans and 5 more fine vans.

The shadow price on a particular constraint represents the change in the value of the objective function per unit increase in the RHS value of that constraint. In case of Joe's van example, the shadow price of capacity limit is $500. We can perform a similar calculation to find the shadow price $60 associated with the labor constraint, implying that an addition of one unit of labor is worth $60.
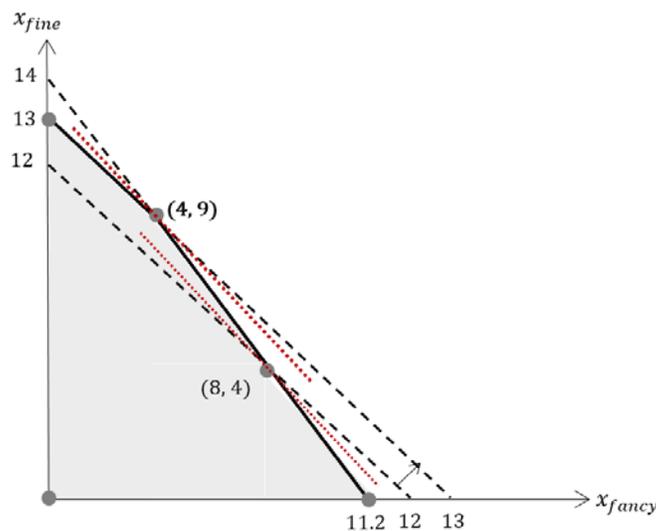
Figure 3-17. One More Units of Capacity Constraint and New Optimal Solution

6	Variable Cells						
7			Final	Reduced	Objective	Allowable	Allowable
8	Cell	Name	Value	Cost	Coefficient	Increase	Decrease
9	$B$4	Production of Fancy	8	0	2000	125	300
10	$C$4	Production of Fine	4	0	1700	300	100
11							
12	Constraints						
13			Final	Shadow	Constraint	Allowable	Allowable
14	Cell	Name	Value	Price	R.H. Side	Increase	Decrease
15	$E$7	Capacity Resouce used	12	500	12	2	0.8
16	$E$8	Labor Resouce used	280	60	280	20	40
17							

Figure 3-18. Sensitivity Report

We may consider the shadow prices associated with the non-negativity constraints. These shadow prices often are called the *reduced costs* and usually are reported separately from the shadow prices on the other constraints; however, they have the identical interpretation. For Joe's van example, the reduced costs are zero, because both fancy and fine vans are already non-negative and increasing either of the non-negativity constraints separately will not affect the optimal solution.

Excel Solver provides ranging RHS analysis. It comes in the form of the second Report options in the Solver Results Box (Figure 3-13) – Sensitivity. Figure 3-18 shows the Sensitivity Report for Joe's van example. Let us focus on the Constraints section of the report, and particularly the column entitled Shadow Price. The shadow price on the capacity limit 500 represents the change in the value of the objective function per unit increase in the RHS value of the capacity (12 to 13). Similarly, Joe would expect $60 more in z when Joe has one more unit of labor. The Variable Cells section reports Reduced Cost of the decision variables, which are zero.

3.4.2.2. *Objective Coefficients Ranges*

The data for a LP may not be known with certainty or may be subject to change. When solving LP, then, it is natural to ask about the sensitivity of the optimal solution to variations in the data. For example, over what range can a particular objective-function coefficient vary without changing the optimal solution? It is clear from Figure 3.5 (Joe's van optimal solution) that some variation of the contribution coefficients is possible without a change in the optimal levels of the decision variables. We will consider first the question of making one-at-a-time changes in the coefficients of the objective function. Figure 3-18, Sensitivity Reports provides the answer with Allowable Increase and Allowable Decrease columns in Variable Cells section.

The objective coefficient, profit margin, for fancy van is $2000 (Objective Coefficient column in Variable Cells section in Figure 3-18). The optimal solutions, $x^*_{fancy} = 8$, $x^*_{fine} = 4$, would not change any profit margin for fancy van between $2125 (= $2000 + 125) and $1700 (= $2000 – 300). Note that z^* would change accordingly because the profit margin changes. Suppose that the profit margin for fancy van is now $2100. The optimal solution (without solving the model) would not change

because it is under the allowable increase. Thus, $x^*_{fancy} = 8$, $x^*_{fine} = 4$, and $z^*_{new} = 23600$.

We can determine the range (allowable increase and decrease) on the objective coefficients for fancy van assuming the remaining coefficients and values in the problem remain unchanged:

(3-10)   Capacity limit slope > Objective slope > Labor limit sope

Since $z = c_{fancy}x_{fancy} + 1700x_{fine}$ can be written as $x_{fine} = \frac{z}{1700} - \left(\frac{c_{fancy}}{1700}\right)x_{fancy}$ (see equation 3-2), then

$$(3\text{-}11) \quad -1 > -\frac{c_{fancy}}{1700} > -\frac{25}{20} \text{ or } 1 < \frac{c_{fancy}}{1700} < 1.25 \ \rightarrow \ 1700 < c_{fancy} < 2125$$

where the current value of $c_{fancy} = 2000$ and allowable increase is 125 (= 2125 – 2000) and allowable decrease is 300 (= 2000 – 1700).

Similarly, by holding $c_{fancy} = 2000$, we can determine the range of c_{fine} as well, which is given by $1600 < c_{fine} < 2000$. The objective ranges are therefore the range over which a particular objective coefficient can be varied, all other coefficients and values in the problem remaining unchanged, and have the optimal solution (i.e., levels of the decision variables) remain unchanged. Note that an optimal solution to a LP is not always unique. If the objective function is parallel to one of the binding constraints, then there is an entire set of optimal solutions. Suppose that Joe's objective function were $z = 1700x_{fancy} + 1700x_{fine}$, i.e., slope = –1, it would be parallel to the capacity limit constraint as shown in Figure 3-19. All levels of the decision variables lying on the line segment joining CPF solution (0, 12) and (8, 4) in Figure 3-19 would be optimal solution.
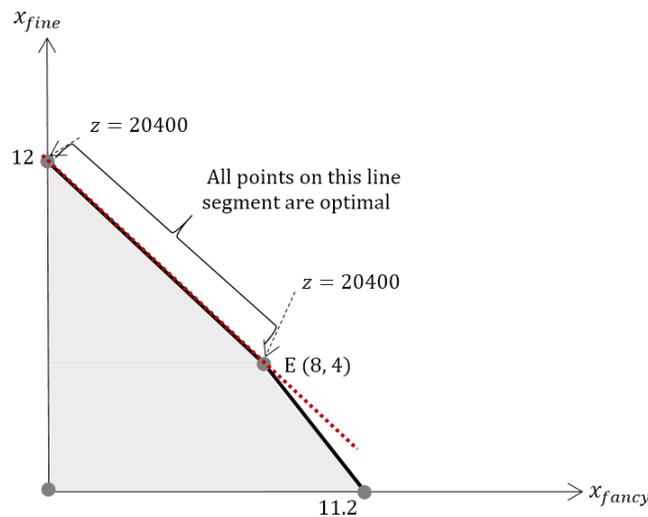
Figure 3-19. Objective Function Coincides with a Constraint

3.4.2.3. *RHS Ranges*

Now consider the question of making one-at-a-time changes in the RHS values of the constraints. Suppose that Joe wants to find the range on the capacity limit. Based on Figure 3-18 Sensitivity Report, allowable increase and decrease column in Constraints section, we know that the capacity limit can be changed up by 2 or down by 0.8. Let's increase the current capacity limit, 12 to 14 (2 units more) and rerun the model. The optimal solution is $x_{fancy} = 0$, $x_{fine} = 14$, and z = 23800, and the shadow price of the constraint is 500. The optimal solution changes but not shadow price. What if capacity limit becomes 15 (beyond allowable increase)? Now the shadow price of the capacity limit becomes zero.

3.5. Duality

Associated with every LP problem is a related *dual* LP problem. The originally formulated LP is known as the *primal* LP problem. If the objective in the primal LP is maximization of a function, then the objective in the dual problem is minimization of related (but different) function. Conversely, a primal minimization problem has a related dual maximization problem. The dual variables represent the variables contained in the dual problem.

3.5.1. *Basic Duality*

The study of duality is very important in LP. Knowledge of duality allows one to develop increased insight into LP solution interpretation. Also, when solving the dual of any problem, one simultaneously solves the primal. Thus, duality is an alternative way of solving LP problems. However, given today's computer capabilities, this is an infrequently used aspect of duality. Therefore, we concentrate on the study of duality as a means of gaining insight into the LP solution.

The (very simple) primal LP is (White Electronic example, 1 = gas, 2 = ele):

$$
\begin{array}{rlrclll}
\max \quad z = & 100x_1 & + & 60x_2 & & & \\
\text{s.t.} \quad & 20x_1 & + & 40x_2 & \leq & 400 & \text{[raw material]} \\
& 5x_1 & + & 2x_2 & \leq & 40 & \text{[machine-processing]} \\
& x_1 & & & \leq & 6 & \text{[division 1 assembly]} \\
& & & x_2 & \leq & 9 & \text{[division 2 assembly]} \\
& x_1 & , & x_2 & \geq & 0 & \text{Non-negativity}
\end{array}
$$

(3-12)

Note that associated with each *constraint* of the primal in equation (3-12) is a dual variable. Because the primal has four constraints (except non-negativity), the dual problem has four variables, say u_1, u_2, u_3, and u_4. The objective of the dual is

(3-13) $\min \quad w = 400u_1 + 40u_2 + 6u_3 + 9u_4$

Similarly, the constraints of the dual problem are associated with x_1 and x_2 such that

(3-14) $20u_1 + 5u_2 + u_3 \geq 100$

$$40u_1 + 2u_2 + u_4 \geq 60$$

with non-negative u_i. Thus, the dual of (3-12) is

$$
\begin{array}{rlrrrrrrll}
\min & w = & 400u_1 & + & 40u_2 & + & 6u_3 & + & 9u_4 & \\
\text{s.t.} & & 20u_1 & + & 5u_2 & + & u_3 & & & \geq \quad 100 \quad (1) \\
(3\text{-}15) & & 40u_1 & + & 2u_2 & & & + & u_4 & \geq \quad 60 \quad (2) \\
& & u_1 & , & u_2 & & u_3 & & u_4 & \geq \quad 0 \quad (3)
\end{array}
$$

In general, if the primal problem has n variables and m resource constraints, the dual problem will have m variables and n constraints. There is a one-to-one correspondence between the primal constraints and the dual variables; i.e., u_1 is associated with the first primal constraint, u_2 with the second primal constraint, etc. As we demonstrate later, dual variables (u_i) can be interpreted as the _marginal value_ (imputed values or shadow prices) of each constraint's resources. These dual variables are usually called shadow prices and indicate the imputed value of each resource. A one-to-one correspondence also exists between the primal variables and the dual constraints; x_1 is associated with the first dual constraint and x_2 is associated with the second dual constraint, etc.

3.5.2. Economic Interpretation

The dual problem economic interpretation is important. The variable u_1 gives the marginal value of the first resource (raw material); u_2 gives the marginal value of the second resource (machine-processing). The first dual constraint restricts the value of the resources used in producing a unit of x_1 (gas dryer) to be greater than or equal to the marginal revenue contribution of x_1. In the primal problem, x_1 uses 20 units of raw materials, 5 units of machine-processing and 1 unit of division 1 assembly, returning \$100, while the dual problem requires raw material use times its marginal value ($20u_1$) plus machine-processing times its marginal value ($5u_2$) plus division 1 assembly times its marginal value (u_3) to be greater than or equal to the profit earned when one unit of x_1 is produced (100). Similarly, the second dual constraint requires the marginal value of resource use ($40u_1 + 2u_2 + u_4$) to be greater than or equal to \$60, which is the amount of profit earned by producing x_2. Thus, the dual variable values are constrained such that the marginal value of the resources used by each primal variable is no less than the marginal profit contribution of that variable.

Now suppose we examine the objective function. This function minimizes the total marginal value of the resources (or minimize the total value of the resource employed in the process, equivalently minimize the total cost). In the example, this amounts to the raw material available times the marginal value of raw material ($400u_1$) plus the machine-processing endowment times the marginal value of the machine-processing ($40u_2$) plus the division 1 and 2 assembly capacity available times their marginal values ($6u_3 + 9u_4$).

Thus, the dual variables arise from a problem minimizing the marginal value of the resource endowment subject to constraints requiring that the marginal value of the resources used in producing each product must be at least as great as the marginal value of the product. This can be

viewed as the problem of a resource purchaser in a perfectly competitive market. Under such circumstances, the purchaser would have to pay at least as much for the resources as the value of the goods produced using those resources. However, the purchaser would try to minimize the total cost of the resources acquired. The resultant dual variable values are measures of the marginal value of the resources. The objective function is the minimum value of the resource endowment.

In short, an important LP theorem, known as the duality, indicates that the maximum value of the primal profit function will also be equal to the minimum value of the dual imputed value function.

3.5.3. Comparison of Solutions

Table 3-1 presents the optimal solutions of equation (3-12) and (3-15) for comparison. As explained, each dual variable (u_i) indicates the rate of change in total profits for an incremental change in the amount of each of the various resources.

For example, $u_2^* = 17.5$ indicates that profits could be increased by as much as $17.50 if an additional unit (hour) of machine capacity could be made available to the production process. This type of information is potentially useful in marking decisions about purchasing or renting additional machine capacity or using existing machine capacity more fully thorough the use of overtime and multiple shifts. A dual variable equal to zero, such as $u_3^* = u_4^* = 0$, indicates that profit would not increase if additional resources of these types were made available; in fact, excess capacity in these resource exists.

Table 3-1: Solution of Primal and Dual

Primal		Dual	
Max obj. fn. z^*	950	Min obj. fn. w^*	950
x_1^*	5.0 dryers	u_1^*	$0.625
x_2^*	7.5 dryers	u_2^*	$17.5
		u_3^*	$0
		u_4^*	$0
Shadow price 1	$0.625	Shadow price 1	5.0 dryers
Shadow price 2	$17.5	Shadow price 2	7.5 dryers
Shadow price 3	$0		
Shadow price 4	$0		

4. LINEAR PROGRAMMING MODELS

Key points:

Standard LP problems are introduced, 1) resource allocation, 2) transportation, 3) diet, feeding or blending, 4) joint production, 5) assembly, and 6) disassembly.

4.1. Introduction

LP formulations are typically composed of a number of standard problem types. We review six basic problems in this chapter.

- Resource allocation problem
- Transportation problem
- Diet/feeding/blending problem
- Joint products problem
- Assembly problem
- Disassembly problem

We will examine i) basic structure, ii) formulation, iii) example application, and iv) solution interpretation.

4.2. Resource Allocation Problem

The classic LP problem involves the allocation of an endowment of scarce resources among a number of competing products so as to maximize profit, for example, Joe's van and White Electronic dryer examples in Chapter 3 are typical resource allocation problems. The key elements of the problem are

- Objective: Maximize profit (or return, sales, revenue)
- Major decision variable x_j is the number of units of the jth product made
- Non negative production $(x_j > 0)$
- Resource usage across all production possibilities is less than or equal to the resource endowment

Algebraic formulation is given by

$$
\begin{aligned}
\max \quad z &= \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \quad \text{for all } i = 1, \cdots, m \\
x_j &\geq 0
\end{aligned}
$$

(4-1)

where c_j is a profit margin per unit of the jth product, a_{ij} is the number of units of the ith resource used when producing one unit of the jth product (technical coefficients), and b_i is the endowment of the ith resource (RHS).

4.2.1. E-Z Chair Makers

Suppose that E-Z Chair Makers are trying to determine how many of each of two types of chairs to produce. Further, suppose that E-Z Chair Makers has four categories of resources which constrain production. These involve the availability of three types of machines: 1) large lathe, 2) small lathe, and 3) chair bottom carver; as well as labor availability. Two types of chairs are produced: functional and fancy. A functional chair costs $15 in basic materials and a fancy chair $25. A finished functional chair sells for $82 and a fancy chair for $105. The resource requirements with the regular method for each product are shown in Table 4.1.

Table 4-1: Resource Requirements for the E-Z Chair Makers for Regular Method

	Hours of use per chair	
	Functional Chair	Fancy Chair
Small lathe	0.8	1.2
Large lathe	0.5	0.7
Chair bottom carver	0.4	1.0
Labor	1.0	0.8

Note that the profit margin for functional chair is $67 (= $82 – $15) and the profit margin for fancy chair is $80 (= $105 – $25). Assume the availability of time is 140 hours for the small lathe, 90 hours for the large lathe, 120 hours for the chair bottom carver, and 125 hours of labor.

Let x_1 = number of functional chair produced and x_2 = number of fancy chair produced, then a formulation is given by equation (4-2):

$$
\begin{array}{rlrlcll}
\max & z = & 67x_1 & + & 80x_2 & & \\
\text{s.t.} & & 0.8x_1 & + & 1.2x_2 & \leq 140 & \text{[Small lathe]} \\
& & 0.5x_1 & + & 0.7x_2 & \leq 90 & \text{[Large lathe]} \\
(4\text{-}2) & & 0.4x_1 & + & 1.0x_2 & \leq 120 & \text{[Bottom carver]} \\
& & 1.0x_1 & + & 0.8x_2 & \leq 125 & \text{[Labor]} \\
& & x_1 & , & x_2 & \geq 0 & \text{[non-negativity]}
\end{array}
$$

There is no difference between equation (4-2) and Joe's van example in equation (3-1). We can solve the model using Excel Solver as we did for Joe's van model. The optimal solution is $x_1^* = 67.86$, $x_2^* = 71.43$, and $z^* = 10260.71$. Note that we assume that x_1 and x_2 are continuous (continuity) and thus we have fractional chairs.

Suppose now that the E-Z Chair Makers has flexibility in the usage of equipment. The chairs may have part of their work substituted between lathes. Labor and material costs are also affected. Data on the substitution possibilities are given in Table 4-2. Assume the availability of time is 140 hours for the small lathe, 90 hours for the large lathe, 120 hours for the chair bottom carver, and 125 hours of labor. Six different chair/production method possibilities can be delineated. Let x_{ij} is the number of chair type i (i = functional (1), fancy (2)) using j production method (j = regular method (1), maximum use of small lathe (2), maximum use of large lathe (3))

x_{11} = number of functional chair made with the regular method

x_{12} = number of functional chair made with the maximum use of the small lathe

x_{13} = number of functional chair made with the maximum use of the large lathe

x_{21} = number of fancy chair made with the regular method

x_{22} = number of fancy chair made with the maximum use of the small lathe

x_{23} = number of fancy chair made with the maximum use of the large lathe

The objective function coefficients require calculation. The basic formula is that profits for the production of x_{ij} (c_{ij}) equal the revenue to the particular type of chair less the relevant base material costs, less any relevant cost increase due to lathe shifts. Thus, c_{11} for x_{11} is calculated by subtracting $15 from $82 (= price of functional chair), yielding the entered $67, and c_{12} for x_{12} is less $1 than $67. Thus,

- Profit for functional chair made with regular method c_{11} = $87 − $15 = $67

- Profit for functional chair made with max use of small lathe c_{12} = $87 − $15 − $1 = $66

- Profit for functional chair made with max use of large lathe c_{13} = $87 − $15 − $0.7 = $66.3

- Profit for fancy chair made with regular method c_{21} = $105 − $25 = $80

- Profit for fancy chair made with max use of small lathe c_{22} = $105 − $25 − $1.5 = $78.5

- Profit for fancy chair made with max use of large lathe c_{23} = $105 − $25 − $1.6 = $78.4

Table 4-2: Resource Requirements and Increases Costs for Alternative Methods to Produce Chairs for the E-Z Chair Makers

	Maximum use of small lathe		Maximum use of large lathe	
	Functional	Fancy	Functional	Fancy
Small lathe	1.30	1.70	0.20	0.50
Large lathe	0.20	0.30	1.30	1.50
Chair bottom carver	0.40	1.00	0.40	1.00
Labor	1.05	0.82	1.10	0.84
Cost increase	$1.00	$1.50	$0.70	$1.60

The constraints on the problem impose the availability of each of the four resources. The technical coefficients are those appearing in Tables 4.1 and 4.2. The resultant LP model is:

$$
\begin{array}{rrcrcrcrcrcrcr}
\max & z = & 67x_{11} & + & 66x_{12} & + & 66.3x_{13} & + & 80x_{21} & + & 78.5x_{22} & + & 78.4x_{23} & \\
\text{s.t.} & & 0.8x_{11} & + & 1.3x_{12} & + & 0.2x_{13} & + & 1.2x_{21} & + & 1.7x_{22} & + & 0.5x_{23} & \leq & 140 \\
& & 0.5x_{11} & + & 0.2x_{12} & + & 1.3x_{13} & + & 0.7x_{21} & + & 0.3x_{22} & + & 1.5x_{23} & \leq & 90 \\
& & 0.4x_{11} & + & 0.4x_{12} & + & 0.4x_{13} & + & x_{21} & + & x_{22} & + & x_{23} & \leq & 120 \\
& & x_{11} & + & 1.05x_{12} & + & 1.1x_{13} & + & 0.8x_{21} & + & 0.82x_{22} & + & 0.84x_{23} & \leq & 125 \\
& & x_{11} & , & x_{12} & , & x_{13} & , & x_{21} & , & x_{22} & , & x_{23} & \geq & 0
\end{array}
$$

(4-3)

4.2.2. Model Solution

Figure 4-1 presents the Excel model for the E-Z Chair Makers with optimal solution. The E-Z Chair Makers produces 62 functional chairs using the regular method, 73 fancy chairs using the regular method and 5 fancy chairs using the maximum use of large lathe to earn profit of $10,417. Figure 4-2 presents the sensitivity report. As shown in Figure 4-2, this production plan exhausts small and large lathe resources as well as labor (positive shadow prices imply that these constraints are binding) but chair bottom carver (shadow price is zero meaning that it is not binding). The shadow prices indicate that one more hour of the small lathe is worth $33.33, one more hour of the large lathe $25.79, and one more hour of labor $27.44 (imputed value of resources). The reduced cost valuation information also shows, for example, that functional chair production with maximum use of a small lathe would cost $11.30 a chair or reduces profit by $11.30/chair. Finally, there is excess capacity of 16.91 hours of chair bottom carving (120 hours (RHS) – 103.09 hours).

	A	B	C	D	E	F	G	H	I	J	K
1	**E-Z Chair Makers**										
2		Functional	Functional	Functional	Fancy	Fancy	Fancy				
3		regular method	max smll lathe	max lg lathe	regular method	max smll lathe	max lg lathe				
4		**X11**	**X12**	**X13**	**X21**	**X22**	**X23**		**MAX**		
5	*Production of*	62.23	0.00	0.00	73.02	0.00	5.18	*Profit*	z		
6	Profit margin	67	66	66.3	80	78.5	78.4		10417.291		
7											Resource
8	Constraints								Resouce used		Available
9	*Small lathe*	0.80	1.30	0.20	1.20	1.70	0.50		140.00	<=	140
10	*Large lathe*	0.50	0.20	1.30	0.70	0.30	1.50		90.00	<=	90
11	*Chair bottom carver*	0.40	0.40	0.40	1.00	1.00	1.00		103.09	<=	120
12	*Labor*	1.00	1.05	1.10	0.80	0.82	0.84		125.00	<=	125
13											

Figure 4-1: EZ Chair Makers LP Model and Optimal Solution

6	Variable Cells						
7			**Final**	**Reduced**	**Objective**	**Allowable**	**Allowable**
8	**Cell**	**Name**	**Value**	**Cost**	**Coefficient**	**Increase**	**Decrease**
9	$B$5	Production of X11	62.2333943	0	67	27.8421053	3.92506596
10	$C$5	Production of X12	0	-11.2990555	66	11.2990555	1E+30
11	$D$5	Production of X13	0	-4.07934186	66.3	4.07934186	1E+30
12	$E$5	Production of X21	73.0195003	0	80	4.42299306	4.97174815
13	$F$5	Production of X22	0	-8.3968312	78.5	8.3968312	1E+30
14	$G$5	Production of X23	5.17976843	0	78.4	14.7056564	4.29115385
15							
16	Constraints						
17			**Final**	**Shadow**	**Constraint**	**Allowable**	**Allowable**
18	**Cell**	**Name**	**Value**	**Price**	**R.H. Side**	**Increase**	**Decrease**
19	$I$9	Small lathe Resouce used	140	33.32723949	140	11.3333333	44.3796296
20	$I$10	Large lathe Resouce used	90	25.78915296	90	22.7045827	6.07142857
21	$I$11	Chair bottom carver Resouce used	103.092626	0	120	1E+30	16.9073736
22	$I$12	Labor Resouce used	125	27.44363193	125	50.4526316	27.0682927

Figure 4-2: EZ Chair Makers LP Model and Optimal Solution

4.3. Transportation Model

The second problem covered is the transportation problem. This problem involves the shipment of a homogeneous product from a number of supply locations to a number of demand locations. Setting this problem up algebraically requires definition of indices for: a) the supply points which we will designate as i, and b) the demand locations which we will designate as j. In turn, the variables indicate the quantity shipped from each supply location to each demand location. We define this set of variables as x_{ij} (the quantity shipped from i to j).

There are three general types of constraints, one allowing only nonnegative shipments, one limiting shipments from each supply point to existing supply and the third imposing a minimum demand requirement at each demand location. Definition of the supply constraint requires specification of the parameter s_i which gives the supply available at point i, as well as the formation of an expression requiring the sum of outgoing shipments from the ith supply point to all possible destinations, j, to not exceed s_i. Algebraically this expression is

$$(4\text{-}4) \quad \sum_j x_{ij} \leq s_i \quad \text{for all i}$$

Definition of the demand constraint requires specification of the demand quantity d_j required at demand point j, as well as the formation of an expression summing incoming shipments to the jth demand point from all possible supply points, i. Algebraically this yields

$$(4\text{-}5) \quad \sum_i x_{ij} \geq d_j \quad \text{for all j}$$

Finally, the objective function depicts minimization of total cost across all possible shipment routes.

This involves definition of a parameter c_{ij} which depicts the cost of shipping one unit from supply point i to demand point j. In turn, the algebraic formulation of the objective function is the first equation in the composite formulation below.

$$\min \quad z = \sum_i \sum_j c_{ij} x_{ij}$$

s.t.

(4-6)

$$\sum_j x_{ij} \leq s_i \quad \text{for all i}$$

$$\sum_i x_{ij} \geq d_j \quad \text{for all j}$$

$$x_{ij} \geq 0 \quad \text{for all i, j}$$

This particular problem is a cost minimization problem rather than a profit maximization problem. The transportation variables, x_{ij}, belong to the general class of transformation variables. Such variables transform the characteristics of a good in terms of form, time, and/or place characteristics. In this case, the transportation variables transform the place characteristics of the good by transporting it from one location to another. The supply constraints are classical resource availability constraints. However the demand constraint imposes a minimum level and constitutes a minimum requirement constraint.

4.3.1. ABC Company

ABC Company has three plants which serve four demand markets. The plants (supply points i) are in New York, Chicago, and Los Angeles. The demand markets (demand points j) are in Miami, Houston, Minneapolis and Portland. The quantity available at each supply point and the quantity required at each demand market are in Table 4-3.

The assumed distances between cities are presented in Table 4-4 in 000 miles (1 mile = 1.609 km). Also assume that the firm has discovered that the cost of moving goods is related to distance (D) by the formula; $c_{ij} = 5 + 5D_{ij}$, where D_{ij} is the distance (in miles) between supply point i and demand point j. Given these distances, the transportation costs are calculated in Table 4-5.

Table 4-3: Quantity Available at Supply Point and Required at Demand Point

Supply available		Demand required	
New York	100	Miami	30
Chicago	75	Houston	75
Los Angeles	90	Minneapolis	90
		Portland	50

Table 4-4: Distances (000 miles) and Transportation Costs between Cities (000 dollars)

Distance	To			
From	Miami	Houston	Minneapolis	Portland
New York	1.29	1.63	1.20	2.89
Chicago	1.38	1.08	0.43	2.12
Los Angeles	2.73	1.55	1.93	0.96
Cost	To			
From	Miami	Houston	Minneapolis	Portland
New York	11.45	13.15	11.00	19.45
Chicago	11.90	10.40	7.15	15.60
Los Angeles	18.65	12.75	14.65	9.80

The above data allow formulation of an LP transportation problem. Let i denote the supply points where i = 1 denotes New York, i = 2 Chicago, and i = 3 Los Angeles. Let j represent the demand points where j = 1 denotes Miami, j = 2 Houston, j = 3 Minneapolis, and j = 4 Portland. Next define x_{ij} as the quantity shipped from city i to city j; e.g., x_{23} stands for the quantity shipped from Chicago to Minneapolis. A formulation of this problem is given in Figure 4-3 in Excel with optimal solution.

Figure 4-3. ABC Company LP Model and Optimal Solution

4.3.2. Model Solution

The optimal value of the objective function value is 2,505 (000 dollars) or 2.5 million dollars. The optimal shipping pattern is shown in Figure 4.3 and Figure 4-4. The solution shows twenty units are left in New York's potential supply (since constraint 1 is in slack). All units from Chicago are exhausted and the marginal value of additional units in Chicago equals $3.85 (negative shadow price, which is the savings realized if more supply were available at Chicago which allowed an increase in

the volume of Chicago shipments to Minneapolis and thereby reducing New York-Minneapolis shipments).

The solution also shows what happens if unused shipping routes are used. For example, the anticipated increase in cost that would be necessary if one were to use the route from New York to Portland (x_{14}) is $9.25 (reduced cost), which would indicate a reshuffling of supply. For example, Los Angeles would reduce its shipping to Portland and increase shipping to somewhere else (probably Houston).

			Final	Reduced	Objective	Allowable	Allowable
	Cell	Name	Value	Cost	Coefficient	Increase	Decrease
9	$B$6	Shipments X11	30	0	11.45	4.3	11.45
10	$C$6	Shipments X12	35	0	13.15	1.1	0.4
11	$D$6	Shipments X13	15	0	11	4.05	1.1
12	$E$6	Shipments X14	0	9.25	19.45	1E+30	9.25
13	$F$6	Shipments X21	0	4.3	11.9	1E+30	4.3
14	$G$6	Shipments X22	0	1.1	10.4	1E+30	1.1
15	$H$6	Shipments X23	75	0	7.15	1.1	1E+30
16	$I$6	Shipments X24	0	9.25	15.6	1E+30	9.25
17	$J$6	Shipments X31	0	7.6	18.65	1E+30	7.6
18	$K$6	Shipments X32	40	0	12.75	0.4	9.25
19	$L$6	Shipments X33	0	4.05	14.65	1E+30	4.05
20	$M$6	Shipments X34	50	0	9.8	9.25	10.2

Row 6: Variable Cells

Row 22: Constraints

			Final	Shadow	Constraint	Allowable	Allowable
	Cell	Name	Value	Price	R.H. Side	Increase	Decrease
25	$N$12	Miami Demand Required Quantity	30	11.45	30	20	30
26	$N$13	Houston Demand Required Quantity	75	13.15	75	20	35
27	$N$14	Minneapolis Demand Required Quantity	90	11	90	20	15
28	$N$15	Portland Demand Required Quantity	50	10.2	50	20	35
29	$N$9	NY Supply Available Quantity	80	0	100	1E+30	20
30	$N$10	Chicago Supply Available Quantity	75	-3.85	75	15	20
31	$N$11	LA Supply Available Quantity	90	-0.4	90	35	20

Figure 4-4. ABC Company LP Model and Optimal Solution

4.3.3. Comments

The transportation problem is a basic component of many LP problems. It has been extended in many ways and has been widely used in applied work. A number of assumptions are contained in the above model. First, transportation costs are assumed to be known and independent of volume. Second, supply and demand are assumed to be known and independent of the price charged for the product. Third, there is unlimited capacity to ship across any particular transportation route. Fourth, the problem deals with a single commodity or an unchanging mix of multiple commodities.

These assumptions have spawned many extensions, including, for example, the transshipment problem, wherein transshipment through intermediate cities is permitted. Another extension allows the quantity supplied and demanded to depend on price. This problem is called a spatial equilibrium model and is covered in the price endogenous modeling chapter. Multi-commodity transportation problems have also been formulated. Cost/volume relationships have been included as in the warehouse location model in the integer programming chapter. Finally, the objective function may be defined as containing more than just transportation costs. Ordinarily one thinks of the problem wherein the c_{ij} is the cost of transporting goods from supply point i to demand point j. However, the supply cost may be included so the overall objective function then involves minimizing delivered cost. Also the transport cost may be defined as the demand price minus the transport cost minus the supply price, thereby converting the problem into a profit maximization problem.

4.4. Diet/Feed Mix/Blending Problem

Another classic problem concerns blending or mixing ingredients to obtain a product with certain characteristics or properties. To understand the basic components of the model let's think about the following simple feed mix problem. Feed mix refers to the process of producing (animal) feed from agricultural and/or raw products. To make the question simple, say we need to determine optimum amounts of three ingredients to include in an animal feed mix. The final product must satisfy several nutrient restrictions. The possible ingredients, their nutritive contents (in kilograms of nutrient per kilograms of ingredient) and the unit cost are shown in the following Table 4-6. The mixture of three ingredients in Table 4-6 must meet the following requirements. Note that one kilogram of the feed is to be mixed.

- Calcium: at least 0.8% but not more than 1.2%
- Protein: at least 22% and no upper limit
- Fiber: no lower limit and at most 5%

Let x_j (j = limestone (lim), corn (crn), soybean (soy)) is the amount of ingredient mixed to produce one kilogram of the feed. Total cost is

(4-7) $z = 10.0x_{lim} + 30.5x_{crn} + 90.0x_{soy}$

The constraints of the problem include the normal nonnegativity restrictions plus three additional constraint types: one for the minimum requirements by nutrient, one for the maximum requirements by nutrient and one for the total volume of the diet. For example, the calcium restriction is

(4-8)
$$0.380x_{lim} + 0.001x_{crn} + 0.002x_{soy} \leq 0.012 \text{ [maximum calcium]}$$
$$0.380x_{lim} + 0.001x_{crn} + 0.002x_{soy} \geq 0.008 \text{ [minimum calcium]}$$

Table 4-5: Ingredient Costs and Nutritive Contents

Ingredient j	Calcium (kg/kg)	Protein (kg.kg)	Fiber (kg/kg)	Unit cost (cents/kg)
Limestone	0.380	-	-	10.0
Corn	0.001	0.09	0.02	30.5
Soybeans	0.002	0.50	0.08	90.0

All together, the LP formulation is given by equation (4-9).

$$
\begin{array}{llllllll}
\min \quad z = & 10.000x_{lim} & + & 30.500x_{crn} & + & 90.000x_{soy} & & \text{[mix cost]} \\
\text{s.t.} & 0.380x_{lim} & + & 0.001x_{crn} & + & 0.002x_{soy} & \leq 0.012 & \text{[max calcium]} \\
& & + & 0.020x_{crn} & + & 0.080x_{soy} & \leq 0.050 & \text{[max fiber]} \\
(4\text{-}9) & 0.380x_{lim} & + & 0.001x_{crn} & + & 0.002x_{soy} & \geq 0.008 & \text{[min calcium]} \\
& & + & 0.090x_{crn} & + & 0.500x_{soy} & \geq 0.022 & \text{[min protein]} \\
& x_{lim} & + & x_{crn} & + & x_{soy} & = 1 & \text{[1 kg of feed]} \\
& x_{lim} & , & x_{crn} & , & x_{soy} & \geq 0 &
\end{array}
$$

The optimal solution is $x_{lim} = 0.028$ kg, $x_{crn} = 0.649$ kg, and $x_{soy} = 0.323$ kg; the minimum cost of mixing feed is $49.16.

The above example involves composing a minimum cost diet from a set of available ingredients while maintaining nutritional characteristics within certain bounds. A total dietary volume constraint is also present. Define index, i, representing the nutritional characteristics, calcium, protein, fiber, etc., which must fall within certain limits. Define index, j, which represents the types of feedstuffs, limestone, corn, soybean, etc., available from which the diet can be composed. Then define a variable, x_j, which represents how much of each feedstuff is used in the diet.

The constraints of the problem as in equation (4-9) include three additional constraint types: one for the minimum requirements by nutrient, one for the maximum requirements by nutrient and one for the total volume of the diet. In setting up the nutrient based constraints parameters are needed which tell how much of each nutrient is present in each feedstuff as well as the dietary minimum and maximum requirements for that nutrient. Thus, let a_{ij} be the amount of the ith nutrient present in one unit of the jth feed ingredient; and let UL_i and LL_i be the maximum (upper limit) and minimum (lower limit) amount of the ith nutrient in the feed. Then the nutrient constraints are formed by summing the nutrients generated from each feedstuff ($a_{ij}x_j$) and requiring these to exceed the dietary minimum and/or be less than the maximum.

The resultant constraints are

$$\sum_j a_{ij}x_j \leq UL_i \quad \text{[upper limit of ith nutrient]}$$

(4-10)

$$\sum_j a_{ij}x_j \geq LL_i \quad \text{[lower limit of ith nutrient]}$$

A constraint is also needed that requires the ingredients in the diet equal the required weight of the diet. Assuming that the weight of the formulated diet and the feedstuffs are the same, this requirement can be written as

(4-11) $$\sum_j x_j \leq 1$$

Finally an objective function must be defined. This involves definition of a parameter for feedstuff cost, c_j, and an equation which sums the total diet cost across all the feedstuffs, i.e., $z = \sum_j c_j x_j$. The resulting LP formulation is

$$
\begin{aligned}
\min \quad z = & \sum_j c_j x_j && \text{[mix cost]} \\
\text{s.t.} \quad & \sum_j a_{ij}x_j \leq UL_i && \text{for all i} \quad \text{[upper limit of ith nutrient]} \\
& \sum_j a_{ij}x_j \geq LL_i && \text{for all i} \quad \text{[lower limit of ith nutrient]} \\
& \sum_j x_j = 1 && \\
& x_j \geq 0 && \text{for all j} \quad \text{[non-negativity]}
\end{aligned}
$$

(4-12)

This formulation depicts a cost minimization problem.

4.4.1. Cattle Feed Example

Suppose that cattle feeding involves lower and upper limits on net energy, digestible protein, fat, vitamin A, calcium, salt and phosphorus. Further, suppose the feed ingredients available are corn, hay, soybeans, urea, dical phosphate, salt and concentrated vitamin A. One kilogram of the feed is to be mixed. The costs of the ingredients per kilogram are shown in Table 4.7. The nutrient requirements are given in Table 4.8. The nutrient requirements give the minimum and maximum amounts of each nutrient in one kilogram of feed. Thus, there must be between 0.071 and 0.130 kg of digestible protein in one kg of feed. The volume of feed mixed must equal one kilogram. The nutrient compositions of one kg of each potential feed are shown in Table 4.9. Figure 4-5 presents the Excel formulation with the optimal solution.

Table 4-6: Ingredient Costs ($/kg)

Ingredient	Cost	Ingredient	Cost
Corn	$0.500	Soybean	$0.300
Dical phosphate	$0.498	Concentrated Vitamin A	$0.286
Alfalfa hay	$0.077	Urea	$0.332
Salt	$0.110		

Table 4-7: Required Nutrient Characteristics per Kilogram of Mixed Feed

Nutrient	Unit	Minimum	Maximum
Net energy	Mega calories	1.34351	
Digestible protein	kg	0.071	0.130
Fat	kg		0.050
Vitamin A	International units	2200	
Salt	kg	0.015	0.020
Calcium	kg	0.0025	0.010
Phosphorus	kg	0.0035	0.012
Weight	kg	1	1

Table 4-8: Nutrient Content per Kilogram of Ingredients (kg/kg)

Nutrient	Corn	Hay	Soybean	Urea	Dical phosphate	Salt	Vitamin A Concentrate
Net energy	1.48	0.49	1.29				
Digestible protein	0.075	0.127	0.438	2.62			
Fat	0.0357	0.022	0.013				
Vitamin A	600	50880	80				2204600
Salt						1	
Calcium	0.0002	0.0125	0.0036		0.2313		
Phosphorus	0.0035	0.0023	0.0075	0.68	0.1865		

4.4.2. Model Solution

As shown in Figure 4-5, we will use 0.755 kg of corn and 0.083 kg of hay, 0.144 kg of soybean, 0.0035 kg of dical phosphate, and 0.015 kg of salt to produce 1 kg of cattle feed. The minimum cost of producing 1 kg of feed is $0.43.

	A	B	C	D	E	F	G	H	I	J	K
1	**Cattle Feeding**										
2									**MIN**		
3		*Corn*	*Hay*	*Soybean*	*Urea*	*Dical*	*Salt*	*Vtmn A*	*Total Mix cost*		
4	*Use of*	**0.7553**	**0.0828**	**0.1435**	**0.0000**	**0.0035**	**0.0150**	**0.0000**	**0.4304**		
5	Cost	0.500	0.077	0.300	0.332	0.498	0.110	0.286			
6	**Constraints**										**RHS**
7	*max Protein*	0.0750	0.1270	0.4380	2.6200				0.13000	<=	0.1300
8	*max Fat*	0.0357	0.0220	0.0130					0.03065	<=	0.0500
9	*max Salt*						1.0000		0.01500	<=	0.0200
10	*max Calcium*	0.0002	0.0125	0.0036		0.2313			0.00250	<=	0.0100
11	*max Phosphorus*	0.0035	0.0023	0.0075	0.6800	0.1865			0.00455	<=	0.0120
12	*min Energy*	1.4800	0.4900	1.2900					1.34351	>=	1.34351
13	*min Protein*	0.0750	0.1270	0.4380	2.6200				0.13000	>=	0.07100
14	*min Vtmn A*	600	50880	80				2204600	4675.05840	>=	2200
15	*min Salt*						1.0000		0.01500	>=	0.01500
16	*min Calcium*	0.0002	0.0125	0.0036		0.2313			0.00250	>=	0.00250
17	*min Phosphorus*	0.0035	0.0023	0.0075	0.6800	0.1865			0.00455	>=	0.00350
18	*Volume*	1	1	1	1	1	1	1	1.00000	=	1

Figure 4-5. Cattle Feed LP Model and Optimal Solution

4.4.3. Comments

There are three assumptions within the feed formulation problem. First, the nutrient requirements are assumed constant and independent of the final product (e.g., livestock) price. Second, the quality of each feed ingredient is known. Third, the diet is assumed to depend on only feed price and nutrients. The feed problem is widely used, especially in formulating feed rations. Animal scientists use the term "ration-balancing", and several software programs have been specifically developed to determine least cost rations.

4.5. Joint Products

Many applied LP models involve production of joint products. An example would be a petroleum cracking operation where production yields multiple products such as oil and naphtha. Other examples include dairy production where production yields both milk and calves, or forestry processing where trees yield sawdust and multiple types of sawn lumber. Here, we present a formulation explicitly dealing with joint products. Key variables in model are i) the amount of each product produced for sale, ii) the production process chosen to produce the products, and iii) the amount of market inputs to purchase.

Let's begin with a very simple oil refinery problem to understand the basic components of the model. Say the oil refinery buys crude old (raw or unprocessed input) and produces gasoline, kerosene, and diesel (multiple products). It has two processes, 1 and 2, and the breakouts (proportional yield) are given in Table 4-9.

Table 4-9: Proportional Yield of Refinery Processes

Product	Process 1	Process 2	Sales price
Gasoline (G)	0.40	0.30	$3.0/unit
Kerosene (K)	0.20	0.25	$2.2/unit
Diesel (D)	0.40	0.45	$2.8/unit
Maximum processing	500 units	500 units	

The cost of refining per unit is $0.80 for process 1 and $0.75 for process2 and suppose that the price of crude oil is $1.8/unit. The objective function is maximizing profit, π, from the sale of each product such that

$$(4\text{-}13) \quad \max \quad \pi = 3.0x_G + 2.2x_K + 2.8x_D - 0.80y_1 - 0.75y_2 - 1.8z_C$$

where x_p $(p = G, K, D)$ defines the total quantity of the pth product, y_j $(j = 1, 2)$ identifies the quantity of jth production possibility (process 1 or 2) with corresponding processing cost and z_C indicates the amount of crude oil purchased.

Other than nonnegativity, three types of constraints are needed. The first relates the quantity sold of each product to the quantity yielded by production (supply-demand balance), that is, the quantity of gasoline from processes 1 and 2 is

$$(4\text{-}14) \quad x_G \leq 0.40y_1 + 0.30y_2 \quad \rightarrow \quad x_G - 0.40y_1 - 0.30y_2 \leq 0$$

Here, demand, production of gasoline, x_G, is required to be less than or equal to supply which is the amount generated across the production alternatives, y_1 and y_2. Further, since production and sales are endogenous, this is written as sales minus production and is less than or equal to zero. We have two more supply-demand balance equations for outputs, Kerosene and Diesel such that

$$(4\text{-}15) \quad \begin{aligned} x_K \leq 0.20y_1 + 025y_2 \quad &\rightarrow \quad x_K - 0.20y_1 - 0.25y_2 \leq 0 \\ x_D \leq 0.40y_1 + 0.45y_2 \quad &\rightarrow \quad x_D - 0.40y_1 - 0.45y_2 \leq 0 \end{aligned}$$

The second type of constraint relates the quantity purchased crude oil (raw input) to the quantity utilized by the production activities. In this example it is given by

$$(4\text{-}16) \quad y_1 + y_2 \leq z_C \quad \rightarrow \quad y_1 + y_2 - z_C \leq 0$$

It is a supply-demand balance for input. The Excel formulation with the optimal solution is presented in Figure 4-6.

	A	B	C	D	E	F	G	H	I	J	K
1	**Oil Refinery**										
2											
3		Gasoline	Kerosene	Diesel	Process 1	Process 2	Crude oil	**MAX**			
4		**XG**	**XK**	**XD**	**Y1**	**Y2**	**ZC**	**Profit**			
5	*Quantity of*	350	225	425	500	500	1000	160.000			
6	Prices and Costs	3.00	2.20	2.80	-0.80	-0.75	-1.80				
7	**Constraints**							Constraints		RHS	
8	*S-D G*	1.00			-0.40	-0.30		0.000	<=	0	S-D G
9	*S-D K*		1.00		-0.20	-0.25		0.000	<=	0	S-D K
10	*S-D D*			1.00	-0.40	-0.45		0.000	<=	0	S-D D
11	*S-D Crd oil*				1.00	1.00	-1.00	0.000	<=	0	S-D Crd oil
12	*Process 1*				1.00			500.000	<=	500	Process 1
13	*Process 2*					1.00		500.000	<=	500	Process 2

Figure 4-6. Oil Refinery LP Model and Optimal Solution

Formulation of this model requires indices which depict the products which could be produced, p, the production possibilities, j, the fixed price inputs purchased, k, and the resources which are available in fixed quantity, m.

Three types of variables need to be defined. The first, x_p, defines the total quantity of the pth product sold; the second, y_j, identifies the quantity of jth production alternatives utilized; and the third, z_k, is the amount of the kth input purchased. As discussed in the above example, other than nonnegativity, three types of constraints are needed. The first relates the quantity sold of each product to the quantity yielded by production. Algebraic specification requires definition of a parameter, q_{pj}, which gives the yield of each product, p, by each production possibility. The expression

$$(4\text{-}17) \quad x_p - \sum_j q_{pj} y_j \leq 0$$

is a supply demand balance. Here demand, in the form of sales of pth product, is required to be less than or equal to supply which is the amount generated across the production alternatives. Further, since production and sales are endogenous, this is written as sales minus production and is less than or equal to zero.

The second type of constraint relates the quantity purchased of each fixed price input to the quantity utilized by the production activities. Let the parameter r_{kj} gives the use of the kth input by the jth production possibility. In turn, the constraint sums up total fixed price input usage and equates it to purchases as follows:

$$(4\text{-}18) \quad \sum_j r_{kj} y_j - z_k \leq 0$$

This constraint is another example of a supply demand balance where the endogenous demand in

this case, use of the kth input, is required to be less than or equal to the endogenous supply which is the amount purchased.

The third type of constraint is a classical resource availability constraint which insures that the quantity used of each fixed quantity input does not exceed the resource endowment. Specification requires definition of parameters for the resource endowment (b_m) and resource use when the production possibility is utilized. The constraint which restricts total resource usage across all possibilities is

$$(4\text{-}19) \quad \sum_j s_{mj} y_j \leq b_m$$

where s_{mj} is the use of the mth resource (technical coefficients) by y_j.

For the objective function, an expression is needed for total profits. To algebraically expressed the profits require parameters for the sales price (c_p), the input purchase cost (e_k), and any other production costs associated with production (d_j). Then the objective function can be written as

$$(4\text{-}20) \quad \sum_p c_p x_p - \sum_j d_j y_j - \sum_k c_k z_k$$

The individual terms do not reflect the profit contribution of each variable in an accounting sense, rather this occurs across the total model. Thus, the production variable term (d_j) does not include either the price of the products sold or the cost of all the inputs purchased, but these components are included by terms on the sales and purchase variables. The resultant composite joint products model is

$$
\begin{aligned}
\max \quad z = \sum_p c_p x_p &- \sum_j d_j y_j - \sum_k c_k z_k \\
\text{s.t.} \quad x_p - \sum_j q_{pj} y_j &\leq 0 \quad \text{for all p} \\
\sum_j r_{kj} y_j - z_k &\leq 0 \quad \text{for all k} \\
\sum_j s_{mj} y_j &\leq b_m \quad \text{for all m} \\
x_p, \quad y_j, \quad z_k &\geq 0 \quad \text{For all p, j, k}
\end{aligned}
$$

(4-21)

Several features of this formulation are worth mention. First, note the explicit joint product relationships. When activity y_j is produced, a mix of joint outputs ($q_{pj}, p = 1,2,\cdots$) is produced while simultaneously consuming the variable inputs both directly priced in the objective function (dj) and explicitly included in constraints (r_{kj}), along with the fixed inputs (s_{mj}). Thus, we have a multi-factor, multi-product production relationship.

Another feature of this problem involves the types of variables and constraints which are used. The variables x_p are sales variables which sell the available quantities of the outputs. The variables z_k are purchase variables which supply the inputs utilized in the production process. The variables y_j are production variables. In this case, the production variables show production explicitly in the matrix, and the product is sold through another activity. The first two constraints are supply-demand balances. The last constraint is a resource endowment.

4.5.1. Wheat and Straw Production

Consider a farm which produces both wheat and wheat straw using seven production processes. The basic data for these production processes are given in Table 4.11. The production process involves the joint production of wheat and straw using land, seed and fertilizer.

The relevant prices are wheat, $4.00 per bushel, wheat straw, $0.50 per bale, seed (cost), $.020/lb., and fertilizer (cost), $2.00 per kilogram. Also there is a $5 per acre production cost for each of the processes and the farm has 500 acres.

Note that p = wheat (wht), straw(stw), $j = 1, \cdots, 7$, and k = fertilizer (fert), seed. Objective function is maximizing profit from the sale of each product:

$$(4\text{-}22) \quad \max \quad z = 4x_{wht} + 0.5x_{stw} - 5y_1 - 5y_2 - 5y_3 - 5y_4 - 5y_5 - 5y_6 - 5y_7 - 2z_{fert} - 0.2z_{seed}$$

The supply-demand balance for outputs (wheat and straw) is given by

$$(4\text{-}23) \quad \begin{array}{l} x_{wht} - 30y_1 - 50y_2 - 65y_3 - 75y_4 - 80y_5 - 80y_6 - 75y_7 \le 0 \quad \text{[wheat balance]} \\ x_{stw} - 10y_1 - 17y_2 - 22y_3 - 26y_4 - 29y_5 - 31y_6 - 32y_7 \le 0 \quad \text{[straw balance]} \end{array}$$

Table 4-10: Data for the Wheat and Straw Example Problem

		Processes							Price or cost
Product		1	2	3	4	5	6	7	
Wheat	Yield in bushel	30	50	65	75	80	80	75	$4.0/unit
Wheat straw	Yield in bales	10	17	22	26	29	31	32	$0.5/unit
Fertilizer usage	in Kg	0	5	10	15	20	25	30	$2.0/unit
Seed		10	10	10	10	10	10	10	$0.2/unit
Land		1	1	1	1	1	1	1	$5.0/unit
									500 acres

In addition there is the supply-demand balance for inputs (fertilizer and seed):

(4-24)
$$0y_1 + 5y_2 + 10y_3 + 15y_4 + 20y_5 + 25y_6 + 30y_7 - z_{fert} \leq 0 \quad \text{[fertilizer balance]}$$
$$10y_1 + 10y_2 + 10y_3 + 10y_4 + 10y_5 + 10y_6 + 10y_7 - z_{seed} \leq 0 \quad \text{[seed balance]}$$

Land constraint is also constructed:

(4-25) $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 \leq 500$ [land]

The Excel formulation with the optimal solution and sensitive reports are presented in Figure 4-7.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	**Wheat production**														
2															
3		Wheat	Straw	Processes							Fertilizer	Seed	**MAX**		
4		**Xwheat**	**Xstraw**	**Y1**	**Y2**	**Y3**	**Y4**	**Y5**	**Y6**	**Y7**	**Zfert**	**Zseed**	**Profit**		
5	Quantity of	40000	14500	0	0	0	0	500	0	0	10000	5000	143750		
6	Prices and Costs	4.0	0.5	-5.0	-5.0	-5.0	-5.0	-5.0	-5.0	-5.0	-2.0	-0.2			
7	**Constraints**												Constraints		RHS
8	Wheat balance	1		-30	-50	-65	-75	-80	-80	-75			0.000 <=		0
9	Straw balance		1	-10	-17	-22	-26	-29	-31	-32			0.000 <=		0
10	Fertilizer balance			5	10	15	20	25	30		-1		0.000 <=		0
11	Seed balance			10	10	10	10	10	10	10		-1	0.000 <=		0
12	Land Constraint			1	1	1	1	1	1	1			500.000 <=		500
13															

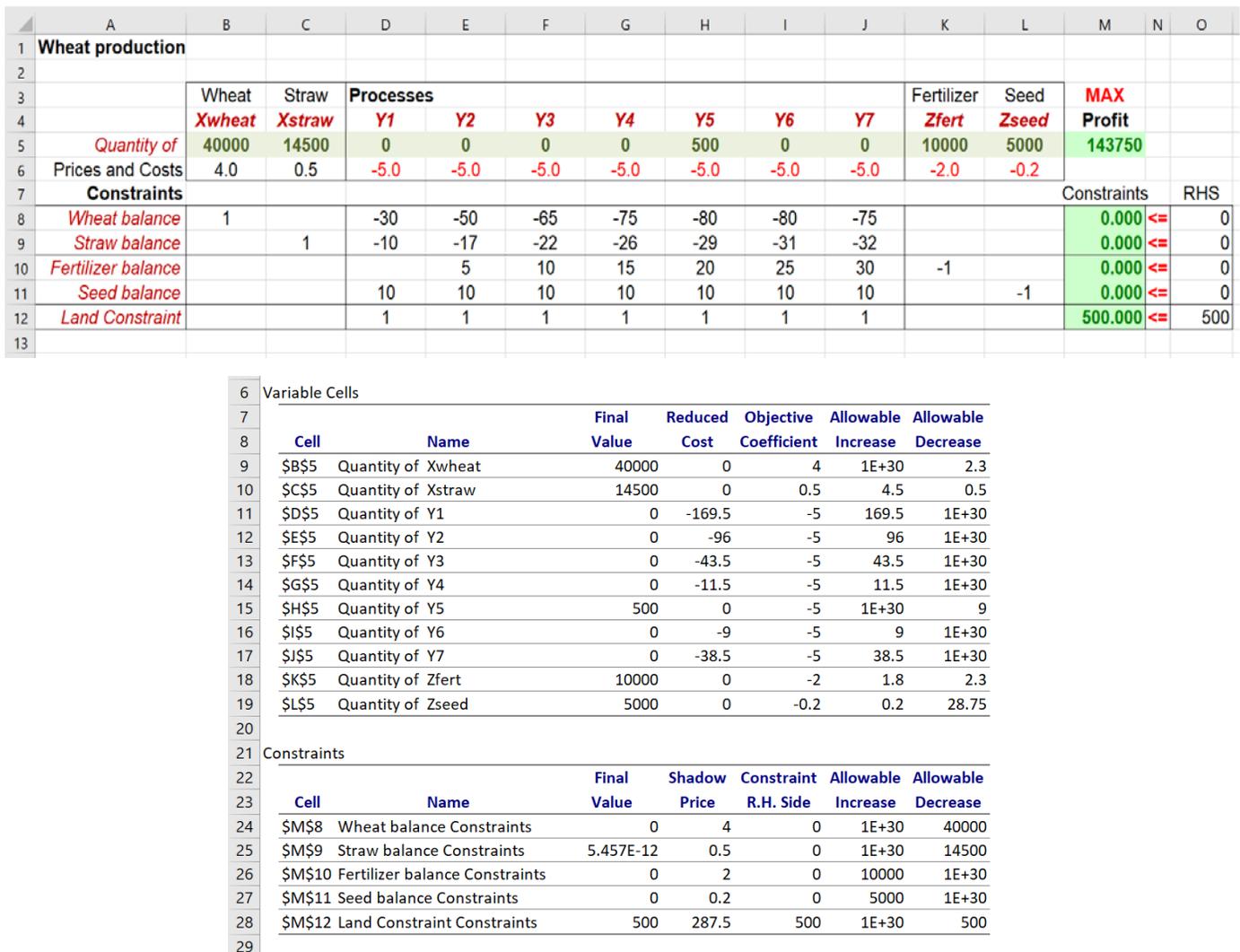| 6	Variable Cells						
7				**Final**	**Reduced**	**Objective**	**Allowable**	**Allowable**
8	**Cell**	**Name**		**Value**	**Cost**	**Coefficient**	**Increase**	**Decrease**
9	$B$5	Quantity of Xwheat		40000	0	4	1E+30	2.3
10	$C$5	Quantity of Xstraw		14500	0	0.5	4.5	0.5
11	$D$5	Quantity of Y1		0	-169.5	-5	169.5	1E+30
12	$E$5	Quantity of Y2		0	-96	-5	96	1E+30
13	$F$5	Quantity of Y3		0	-43.5	-5	43.5	1E+30
14	$G$5	Quantity of Y4		0	-11.5	-5	11.5	1E+30
15	$H$5	Quantity of Y5		500	0	-5	1E+30	9
16	$I$5	Quantity of Y6		0	-9	-5	9	1E+30
17	$J$5	Quantity of Y7		0	-38.5	-5	38.5	1E+30
18	$K$5	Quantity of Zfert		10000	0	-2	1.8	2.3
19	$L$5	Quantity of Zseed		5000	0	-0.2	0.2	28.75
20								
21	Constraints							
22				**Final**	**Shadow**	**Constraint**	**Allowable**	**Allowable**
23	**Cell**	**Name**		**Value**	**Price**	**R.H. Side**	**Increase**	**Decrease**
24	$M$8	Wheat balance Constraints		0	4	0	1E+30	40000
25	$M$9	Straw balance Constraints		5.457E-12	0.5	0	1E+30	14500
26	$M$10	Fertilizer balance Constraints		0	2	0	10000	1E+30
27	$M$11	Seed balance Constraints		0	0.2	0	5000	1E+30
28	$M$12	Land Constraint Constraints		500	287.5	500	1E+30	500
29								

Figure 4-7. Wheat and Straw Joint Production Formulation, Optimal Solution, and Sensitive Report

4.5.2. Model Solution

As shown in Figure 4.7, 40,000 bushels of wheat and 14,500 bales of straw are produced by 500 acres of the fifth production possibility (y_5) using 10,000 kilograms of fertilizer and 5,000 lbs. of seed. The reduced cost information shows a $169.50 cost for the first production possibility if undertaken. Under this production pattern, the marginal value of land is $287.50. The shadow prices on the first four rows are the sale and purchase prices of the various outputs and inputs depicted in those rows.

4.5.3. Comments

The joint products problem illustrates: 1) the proper handling of joint products and 2) production variables where the returns from production are not collapsed into the objective function but explicitly appear in the constraints.

The formulation also illustrates the possible complexity of LP. In this case product balance constraints are incorporated in a model along with resource constraints. Also note that x_{wht} and x_{stw}, give the sum of total output, and that z_{fert} and z_{seed} give the sum of total input usage on the farm which may be convenient for model interpretation.

Joint product formulations have a relatively long history. It is difficult to cite many exact applications; rather such a structure is common and implicit in many models throughout the literature.

4.6. Assembly Problem

An important LP formulation involves the assembly (assembling final products) or blending problem. This problem deals with maximizing profit when assembling final products from component parts. The problem resembles the feed formulation problem where mixed feeds are assembled from raw commodities; however, the assumption of known component mixtures is made. As we did in the above sections, let's discuss a simple LP problem to understand the basic components of the model.

Say Joe's Diner makes two kinds of burger, a cheeseburger and double cheeseburger for sale (final products are cheeseburger and double cheeseburger). Components required to make a burger are presented in Table 4-11.

Let x_1 = cheeseburger and x_2 = double cheeseburger (outputs) and q_1 = bun, q_2 = cheese, q_3 = ground beef; the LP formulation is given by equation 4-26:

Table 4-11: Component Required to Make a Burger

	Cheese burger	Double burger	Cost
Bun	1	1	$0.5/bun
Cheese (slice)	1	2	$0.5/slice
Ground beef (lb.)	0.25	0.50	$2.0/lb.
Labor (hours)	0.25	0.25	Joe has 40 hours/day
Sales price	$5	$7	

$$\begin{array}{llllllll}
\text{max} & z = & 5x_1 & + & 7x_2 & - & 0.5q_1 & - & 0.5q_2 & - & 2q_3 \\
\text{s.t.} & & x_1 & + & x_2 & - & q_1 & & & & & \leq & 0 \\
& & x_1 & + & 2x_2 & & & - & q_2 & & & \leq & 0 \\
& & 0.25x_1 & + & 0.50x_2 & & & & & - & q_3 & \leq & 0 \\
& & 0.25x_1 & + & 0.25x_2 & & & & & & & \leq & 40 \\
& & x_1 & , & x_2 & , & q_1 & , & q_2 & , & q_3 & \geq & 0
\end{array}$$

(4-26)

where the first constraint, $x_1 + x_2 - q_1 \leq 0$, is the bun supply-balance balance; the second constraint $x_1 + 2x_2 - q_2 \leq 0$, is the cheese supply-demand balance; the third constraint $0.25x_1 + 0.50x_2 - q_3 \leq 0$, is the beef supply-demand balance. Excel formulation with the optimal solution is presented in Figure 4-8.

	A	B	C	D	E	F	G	H	I
1	Joe's Diner								
2									
3		Bugers		Inputs (Qk)					
4		Cheese	Double	Bun	Cheese	Beef	MAX		
5		X1	X2	Q1	Q2	Q3	Profit		
6	Quantity of						0		
7	Prices and Costs	5.0	7.0	-0.5	-0.5	-2.0			
8	Constraints						Constraints		RHS
9	Bun balance	1	1	-1			0.000	<=	0
10	Cheese balance	1	2		-1		0.000	<=	0
11	Beef balance	0.25	0.50			-1	0.000	<=	0
12	Labor	0.25	0.25				0.000	<=	40

Figure 4-8. Joe's Diner – Assembly and Optimal Solution

The problem formulation involves k component parts (q_k), i.e., bun, cheese, and beef, which can be purchased at a fixed price. The decision maker is assumed to maximize the value of the final products (x_j), i.e., burgers, assembled. Each of the final products uses component parts via a known formula. Also, fixed resources constrain the production of final products and the purchase of component parts. The formulation is

$$\text{max} \quad z = \sum_j c_j x_j - \sum_k d_k q_k$$

(4-27)

$$\text{s.t.} \quad \sum_j a_{kj} x_j - w_k q_k \leq h_k \quad \text{for all k}$$

$$\sum_j e_{ij} x_j + \sum_k f_{ik} q_k \leq b_i \quad \text{for all i}$$

$$x_j \quad , \quad q_k \geq 0 \quad \text{[non-negative]}$$

where j is the final product index; c_j is the return (or price) per unit of final product j assembled; x_j is the number of units of final product j assembled; k is the component part index; d_k is the cost per unit of component part k; q_k is the quantity of component part k purchased; a_{kj} is the quantity of component part k used in assembling one unit of product j; w_k is the number of units of the component part received when q_k is purchased; i is the index on resource limits (such as labor); e_{ij} is the use of limited resource i in assembling one unit of product j; f_{ik} is the use of the ith limited resource when acquiring one unit of q_k; b_i is the amount of limited resource i available; and h_k is the firm's inventory of ingredient k if any (otherwise 0).

In this formulation the objective function maximizes the return summed over all the final products produced less the cost of the component parts purchased. The first constraint equation is a supply-demand balance and constrains the usage of the component parts to be less than or equal to inventory plus purchases. The second constraint limits the resources used in manufacturing final products and purchasing component parts to the exogenous resource endowment. All of the variables are assumed to be nonnegative.

4.6.1. *Express Computer*

Express Computer assembles six different laptop computer types: R series 14R, 15R, and 17R; Z series 14Z, 15Z and 17Z. Each different type of computer requires a specific set of component parts as shown in Table 4-12. Table 4-13 contains component parts' prices, inventory, and resource (labor and shelf space) requirements. Final products assembly and sales information are in Table 4-15. Note that profit margin is the difference between sales price and assembly cost, for example, profit margin for 14R is given by \$630 = \$689 – \$59. Excel formulation with the optimal solution is presented in Figure 4-9.

Table 4-12: Component Required to Assemble a Laptop

	R series			Z series		
	14R	15R	17R	14Z	15Z	17Z
DVD+RW	1	1		1		
Blu-Ray Disc			1		1	1
Processor 2.5 GHz	1	1		1		
Processor 3.1 GHz			1		1	1
500 GB HD	1	1	1			
750 GB HD				1	1	1
Plain Case	1	1	1			
Fancy Case				1	1	1

Table 4-13: Component Part Acquisition Information

	Cost in $	Inventory	Labor	Shelf space
DVD+RW	35	20	0.01	0.01
Blu-Ray Disc	49	29	0.01	0.01
Processor 2.5 GHz	52	32	0.01	0.01
Processor 3.1 GHz	245	45	0.03	0.03
500 GB HD	102	15	0.07	1.50
750 GB HD	302	45	0.10	2.00
Plain Case	41	11	0.15	1.70
Fancy Case	80	12	0.12	1.70
Limit			550	590

Table 4-14: Final Products Assembly and Sales Information

	Sales price	Min Sales	Assembly cost	Labor	Laptop space
14R	689	1	59	2.00	1
15R	992	3	102	2.05	1
17R	1200	2	100	2.21	1
14Z	1400	4	300	2.24	1
15Z	1500	2	300	2.18	1
17Z	1800	2	400	2.12	1
Limit				550	240

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
2		Assembly Xj						Buy Components Qk								MAX		
3		14R	15R	17R	14Z	15Z	17Z	CD/DVD	Blu-Ray	2.5 GHz	3.1 GHz	500GB HD	750GB HD	Plain	Fancy	MAX		
4		1	160	34	4	2	9	145	16	133	0	180	0	184	3	Profit		
5	Profit Margion	630	890	1100	1100	1200	1400	-35	-49	-52	-245	-102	-302	-41	-80	160739		
6																Used		RHS
7	CD/DVD	1	1		1			-1								20.00	<=	20
8	Blu-Ray			1		1	1		-1							29.00	<=	29
9	2.5 GHz	1	1		1					-1						32.00	<=	32
10	3.1 GHz			1		1	1				-1					45.00	<=	45
11	500GB HD	1	1	1								-1				15.00	<=	15
12	750GB HD				1	1	1						-1			15.00	<=	15
13	Plain	1	1	1										-1		11.00	<=	11
14	Fancy				1	1	1								-1	12.00	<=	12
15	Labor	2	2.05	2.21	2.24	2.18	2.12	0.01	0.01	0.01	0.03	0.07	0.1	0.15	0.12	480.44	<=	550
16	Shelf space							0.01	0.01	0.01	0.03	1.5	2	1.7	1.7	590.00	<=	590
17	Laptop space	1	1	1	1	1	1									209.74	<=	240
18	Lower bound	1														1.00	>=	1
19			1													159.74	>=	3
20				1												34.00	>=	2
21					1											4.00	>=	4
22						1										2.00	>=	2
23							1									9.00	>=	2

Figure 4-9. Express Computer - Assembly

4.6.2. Comments

The assembly problem is related to the feed formulation problem. Namely, the assembly problem assumes that known least cost mixes have been established, and that one wishes to obtain a maximum profit combination of these mixes. There are numerous assumptions in this problem. For example we assume all prices are constant and the quantity of fixed resources is constant. One could extend the model to relax such assumptions.

4.7. Disassembly Problems

Another common LP formulation involves raw product disassembly. This problem is common in agricultural processing where animals are purchased, slaughtered and cut into parts (steak, hamburger, etc.) which are sold. The problem is also common in the forest products and petroleum industries, where the trim, cutting stock and cracking problems have arisen. In the disassembly problem, a maximum profit scheme for cutting up raw products is devised. The primal formulation involves the maximization of the component parts revenue less the raw product purchase costs, subject to restrictions that relate the amount of component parts to the amount of raw products disassembled.

Let's consider a simple example to understand the general formulation. Say a firm buys a car, for example a salvage car, disassembles it into parts and sell the parts. The parts for sale are metal, seats and others. The car can be purchased at $1000, which weighs 2300 pounds. The cost of disassembling the car is $100 and the firm may purchase 10 cars. To disassemble a car, 1 unit of shop capacity (the firm has 10 units) and 10 hours or labor (the firm has 200 hours) required. Proportion breakdown of car into parts and sales prices are presented in Table 4-15.

Let x_j = car (raw material) and q_k = parts (output), k= metal (m), seat (s) and others (o). The total cost is the sum of purchasing a car and disassembling cost, that is, $1100 = $1000 + $100. Thus the objective function is

(4-28)   max $z = -(1000 + 100)x_1 + q_m + 0.9q_s + 0.2q_o$

Metal supply-demand balance is given by proportion breakdown × car weight or $1380 = 0.6 \times 2300$,

(4-29) $-(0.6 \cdot 2300)x_1 + q_m \leq 0$ [metal balance]

We have seats and others supply-demand balance equations as well.

(4-30)
$-(0.2 \cdot 2300)x_1 + q_s \leq 0$ [seat balance]

$-(0.2 \cdot 2300)x_1 + q_o \leq 0$ [others balance]

Table 4-15: Proportion Breakdown of Cars into Parts and Sales Prices

Parts		Part price (sales)	Labor
Metal	0.6	$1.0/pound	0.5
Seats	0.2	$0.9/pound	0.6
Others	0.2	$0.2/pound	0.4

Resource constraints are

$$(4\text{-}31) \quad \begin{aligned} x_1 &\leq 10 & \text{[capacity]} \\ 10x_1 + 0.5q_m + 0.6q_s + 0.4q_o &\leq 100 & \text{[labor]} \end{aligned}$$

All told,

$$(4\text{-}32)$$

$$\begin{array}{rlrrrrrll}
\max \quad z = & -1100x_1 & + & q_m & + & 0.9q_s & + & 0.2q_o & & \\
\text{s.t.} & -1380x_1 & + & q_m & & & & & \leq & 0 \quad \text{[metals balance]} \\
& -460x_1 & & & + & q_s & & & \leq & 0 \quad \text{[seats balance]} \\
& -460x_1 & & & & & + & q_o & \leq & 0 \quad \text{[others balance]} \\
& x_1 & & & & & & & \leq & 10 \quad \text{[capacity]} \\
& 10x_1 & + & 0.5q_m & + & 0.6q_s & + & 0.4q_o & \leq & 200 \quad \text{[labor]} \\
& x_1 & & & & & & & \leq & 10 \quad \text{[upper limit]} \\
& x_1 & , & q_m & , & q_s & , & q_o & \geq & 0 \quad \text{[non-negative]}
\end{array}$$

Based on the example we may have the basic disassembly formulation as follows

$$(4\text{-}33)$$

$$\begin{array}{rlll}
\max \quad z = & -\sum_j c_j x_j & + \sum_k d_k q_k & \\
\text{s.t.} & -\sum_j a_{kj} x_j & - \quad q_k \leq h_k & \text{for all } k \\
& \sum_j e_{ij} x_j & + \sum_k f_{ik} q_k \leq b_i & \text{for all } i \\
& x_j \quad , & q_k \geq 0 & \text{[non-negative]}
\end{array}$$

where j indexes the raw materials, i.e., cars, disassembled; k indexes the component parts sold, i.e., metals, seats and others; i indexes resource availability limits, i.e., capacity and labor; c_j is the cost of purchasing and disassembling one unit of raw product j; x_j is the number of units of raw material j purchased; d_k is the selling price of component part k; q_k is the quantity of component part k sold; a_{kj} is the yield of component part k from raw product j (proportion breakdown); e_{ij} is the use of resource limit i when disassembling raw product j; f_{ik} is the amount of resource limit i used by the sale of one unit of component part k; b_i is the maximum amount of raw product limit i available. We

may have upper and/or lower limits, if any.

The objective function maximizes operating profit, which the sum over all final products sold (q_k) of the total revenue earned by sales less the costs of all purchased inputs. The first constraint is a product balance limiting the quantity sold to be no greater than the quantity supplied when the raw product is disassembled. The next constraint is a resource limitation constraint on raw product disassembly and product sale.

The x_j are production variables indicating the amount of the jth raw product which is disassembled into the component parts (the items produced) while using the inputs e_{ij}. The q_k are sales variables indicating the quantity of the kth product which is sold.

4.7.1. Jerimiah's Junk Yard

The disassembly problem example involves operations at Jerimiah's Junk Yard. The firm is assumed to disassemble up to four different types of cars: Escorts, 626s, T-birds, and Caddy's. Each different type of car yields a unique mix of component parts. The parts considered are metal, seats, chrome, doors and junk. The component part yields from each type of car are given in Table 4-16 as are data on car purchase price, weight, disassembly cost, availability, junk yard capacity, labor requirements, component part minimum and maximum sales possibilities, parts space use, labor use, and sales price.

The resource endowment for labor is 700 hours while there is 42 units of junk yard capacity and 5000 units of parts space. We also extend the basic problem by requiring parts to be transformed to other usages if their maximum sales possibilities have been exceeded. Under such a case, chrome is transformed to metal on a pound per pound basis, while seats become junk on a pound per pound basis, and doors become 70% metal and 30% junk. The problem formulation in Excel is given in Figure 4-10 with the solution.

Table 4-16: Operation Data for Jerimiah's Junk Yard

Raw materials

	Escort	626S	TBIRD	Caddies
Purchase price	85	90	115	140
Weight (lb)	2300	2200	3200	3900
Disassembly cost	100	120	150	170
Availability	13	12	20	10

Resource use

	Escort	626S	TBIRD	Caddies
Capacity	1	1	1.2	1.4
Labor	10	12	15	20

Table 4-16, continued

Proportion breakdown of cars into parts and price

Parts	Escort	626S	TBIRD	Caddies
Metal	0.60	0.55	0.60	0.62
Seats	0.10	0.10	0.06	0.04
Chrome	0.05	0.05	0.09	0.14
Doors	0.08	0.10	0.10	0.07
Junk	0.17	0.20	0.15	0.13

Part sales data

Parts	Min. sales	Max. sales	Part price	Part space	Labor
Metal	0		0.15	0.062	0.0010
Seats	4000	6000	0.90	0.004	0.0015
Chrome	70	10000	0.70	0.014	0.0020
Doors	2	5000	1.00	0.007	0.0025
Junk			-0.05	0.013	0.0001

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	Jeremiah's Junk Yard															
2		RAW MATERIALS				COMPONENTS										
3		E			C											
4		S			A			C			C	C	C			
5		C		T	D	M	S	H	D		N	N	N			
6		O	6	B	D	E	E	R	O	J	V	V	V			
7		R	2	I	I	T	A	O	O	U	R	R	R			
8		T	6	R	E	A	T	M	R	N	T	T	T			
9		S	S	D	S	L	S	E	S	K	SEATS	CHROME	DOORS			
10		Xecrt	X626s	Xtbrd	Xcadd	Qm	Qs	Qc	Qd	Qj	Zs	Zc	Zd	**MAX**		
11		4	0	20	10	73186	6000	10000	5000	18014	320	1680	4866	18337		
12	Objective	-185	-210	-265	-310	0.15	0.90	0.70	1.00	-0.05						
13														Used		RHS
14	Metal	-1380	-1210	-1920	-2418	1						-1	-0.7	0.000	=	0
15	Seats	-230	-220	-192	-156		1				1			0.000	=	0
16	Chrome	-115	-110	-288	-546			1				1		0.000	=	0
17	Doors	-184	-220	-320	-273				1				1	0.000	=	0
18	Junk	-391	-440	-480	-507					1	-1		-0.3	0.000	=	0
19	Capacity	1.0	1.0	1.2	1.4									42.0	<=	42
20	Labor	10	12	15	20	0.0010	0.0015	0.0020	0.0025	0.0001				656.5	<=	700
21	Part Space					0.062	0.004	0.014	0.007	0.013				4970.7	<=	5000
22	Escort upper	1												4.0	<=	13
23	626 upper		1											0.0	<=	12
24	Tbird upper			1										20.0	<=	20
25	Caddies upper				1									10.0	<=	10
26	Seats max						1							6000.0	<=	6000
27	Chrome max							1						10000.0	<=	10000
28	Door max								1					5000.0	<=	5000
29	Door min								1					5000.0	>=	2
30	Seat min						1							6000.0	>=	4000
31	Chrome min							1						10000.0	>=	70

Figure 4-10. Jeremiah's Junk Yard - Disassembly

4.7.2. Comments

It is difficult to find exact examples of the disassembly problem in literature. This formulation is a rather obvious application of LP which, while having been studied a number of times, is not formally recognized.

5. APPLIED INTEGER PROGRAMMING

Key points:

LP continuity assumption is relaxed; Integer problems arise frequently because some or all of the decision variables must be restricted to integer values.

Yes-or-no decision and logical conditions are formulated using 0-1 indicator (binary) variable, which is widely used in applied works.

5.1. Introduction

LP assumes continuity of the solution region. LP decision variables can equal whole numbers or any other real number (3 or 4 as well as 3.49). However, fractional solutions are not always acceptable. Particular items may only make sense when purchased in whole units (e.g., tractors or airplanes). Integer programming (IP) requires a subset of the decision variables to take on integer values. Usually IP problems involves optimization of a linear objective function subject to linear constraints, non-negativity conditions and integer value conditions. IP also permits modeling of fixed costs, logical conditions, and discrete levels of resources.

The integer valued variables are called integer variables. Problems containing integer variables fall into several classes. A problem in which all variables are integer is a pure integer IP problem. A problem with some integer and some continuous variables, is a mixed-integer IP problem (MIP). A problem in which the integer variables are restricted to equal either zero or one is called a zero-one IP problem. There are pure zero-one IP problems where all variables are zero-one and mixed zero-one IP problems containing both zero-one and continuous variables.

Let's recall Joe's Van example in Chapter 3 (equation 3-1).

$$(5\text{-}1) \quad \begin{array}{rlrclcll} \max \quad z = & 2000x_{fancy} & + & 1700x_{fine} & & & & \\ \text{s.t.} & x_{fancy} & + & x_{fine} & \leq & 12 & \text{[capacity constraint]} \\ & 25x_{fancy} & + & 20x_{fine} & \leq & 280 & \text{[labor constraint]} \\ & x_{fancy} & , & x_{fine} & \geq & 0 & \text{[non-negativity]} \end{array}$$

Both x_{fancy} and x_{fine} are assumed to be a real number (continuous number). However, for example, 11.2 of x_{fine} is not actually acceptable. Joe cannot produce 11.2 fine vans. We may add integer constraints to the question, and, in turn, equation (5-1) becomes

$$(5\text{-}2) \quad \begin{array}{rlrclcll} \max \quad z = & 2000x_{fancy} & + & 1700x_{fine} & & & & \\ \text{s.t.} & x_{fancy} & + & x_{fine} & \leq & 12 & \text{[capacity constraint]} \\ & 25x_{fancy} & + & 20x_{fine} & \leq & 280 & \text{[labor constraint]} \\ & x_{fancy} & , & x_{fine} & \geq & 0 & \text{[non-negativity]} \\ & {\color{red}x_{fancy}} & , & {\color{red}x_{fine}} & & {\color{red}\text{integer}} & \text{[integer restriction]} \end{array}$$

Figure 5-1. Imposing Integer Constraints

In Excel, we impose the integer constraints in Solver Window. Click Add button in Constraints box and select changing cells as in Figure 5-1. Then select "int" in the Add Constraint window to declare these are integers (Add Constraint box in Figure 5-1). The final Solver Parameters box is presented in Figure 5-1 as well.

The optimal solution won't change because they are integer already, $\left(x_{fancy}^*, x_{fine}^*\right) = (8, 4)$ but the LP has now integer constraints (Figure 5-1).

5.2. Feasible Region Characteristic and Solution

Let's consider the following LP problem.

$$
\begin{array}{rrrrcl}
\max \quad z = & 5x_1 & + & x_2 & & \\
\text{s.t.} & -x_1 & + & 2x_2 & \leq & 4 \\
(5\text{-}3) & x_1 & - & x_2 & \leq & 1 \\
& 4x_1 & + & x_2 & \leq & 12 \\
& x_1 & , & x_2 & \geq & 0
\end{array}
$$

The optimal solution is $(x_1^*, x_2^*) = (2.6, 1.6)$, and $z^* = 14.6$ assuming x_1 and x_2 are continuous (Panel A in Figure 5-3). Note that the feasible region is a grey area surrounded by three constraints and non-negativity; there are five corner point feasible (CPF) solutions, $(0,0)$, $(1,0)$, $(2.6,1.6)$, $(2.\bar{2}, 3.\bar{1})$, and $(0, 2)$. As discussed, the optimal solution occurs one of CPF solutions, in this case it is $(2.6, 1.6)$ (Panel A, Figure 5-3).

When x_1 and x_2 are integer, however, the feasible region is not an area, instead, we have dots (combinations of integer x_1 and x_2 as shown in Panel B, Figure 5-3). As shown in Panel B, the optimal solution now is $(x_1^*, x_2^*) = (2, 3)$, and $z^* = 13$. Rounding the solution of continuous x's, $(x_1^*, x_2^*) = (2.6, 1.6)$, to the nearest integer (up or down) is <u>not</u> the optimal solution. As shown in Panel B in Figure 5-2, the optimal solution is **not** on the constraint boundaries (one of CPF solutions) and it is much less. Because the optimal solution is not one of CPF solutions, the Simplex algorithm doesn't work for IP problem. Actually the IP problems are notoriously difficult to solve because it has an unknown number of possible solutions and no general statement can be made about the location of the solution.

Panel A. Continuous x's   Panel B. Integer x's
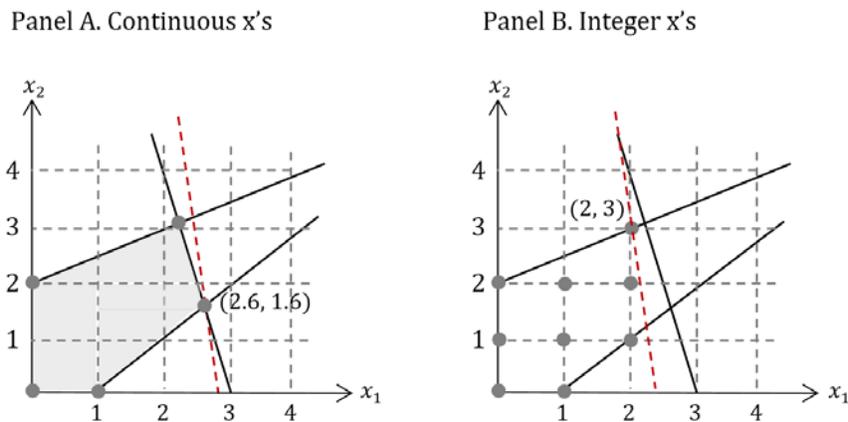
Figure 5-2. Graphical Solution for LP in Equation (5-3)

Dots in Panel A are corner point feasible (CPF) solutions; when an optimal solution of a LP exists, it occurs one of CPF solutions, which is (2.6, 1.6).

5.3. Yes-or-No Decisions

Integer programming may involve a number of interrelated "yes-or-no decisions." In such decisions, the only two possible choices are yes and no or on and off. Binary choice permits modeling of fixed costs (investment) and logical conditions. With just two choices, a decision variable takes just two values, say 0 and 1 (binary variables). Consequently, IP problems that contain only binary variables are called binary integer programming (BIP) model.

The capital budgeting also known as the knapsack problem or cargo loading problem, is a famous BIP

formulation. The knapsack context refers to a hiker selecting the most valuable items to carry, subject to a weight or capacity limit. Partial items are not allowed, thus choices are depicted by zero-one variables. The capital budgeting context involves selection of the most valuable investments from a set of available, but indivisible, investments subject to limited capital availability. The cargo loading context involves maximization of cargo value subject to hold capacity and indivisibility restrictions.

The general formulation (assuming only one of each item is available) is

$$\max \quad z = \sum_j v_j x_j$$

$$\text{(5-4)} \quad \text{s.t.} \quad \sum_j d_j x_j \leq w$$

$$x_j = 0 \text{ or } 1 \quad \text{for all } j$$

The decision variables indicate whether the jth alternative item is chosen ($x_j = 1$) or not ($x_j = 0$). Each item is worth v_j or return from the investment is v_j. The objective function gives the total value of all items chosen. The capacity used by each x_j is d_j or investment costs. The constraint requires total capacity use to be less than or equal to the capacity limit (w).

5.3.1. Thief (Knapsack Problem)

A thief breaks into a house. Around the thief are various objects: a diamond ring, a silver candelabra, a Bose Wave Radio, a large portrait of Elvis Presley painted on a black velvet background, and a large tiffany crystal vase. The thief has a knapsack that can only hold a certain capacity; it can hold a total size of 8. Each of the items has a value and a size (Table 5-1), and cannot hold all of the items in the knapsack.

Table 5-1: Items to be taken

Variable	Items	Volume (Size)	Value (dollars)
x_1	Ring	1	1500
x_2	Candelabra	5	1000
x_3	Bose Radio	3	900
x_4	Elvis portrait	4	500
x_5	Cristal vase	4	400

The resultant BIP formulation is

$$
\begin{array}{llllllllll}
\max & z = & 1500x_1 & + & 1000x_2 & + & 900x_3 & + & 500x_4 & + & 400x_5 \\
\text{(5-5)} & \text{s.t.} & x_1 & + & 5x_2 & + & 3x_2 & + & 4x_4 & + & 4x_5 & \leq & 8 \\
& & & & x_j & = & 0 & & \text{or} & & 1
\end{array}
$$

In Excel, we can impose the binary choice Solver Window. Click Add button in Constraints box and select changing cells as in Figure 5-1. Then select "bin" in the Add Constraint window to declare these are binary variables. The Excel formulation with the optimal solution is presented in Figure 5-4, and take ring, radio and Elvis portrait.

Figure 5-3. Graphical Solution for LP in Equation (5-3)

5.3.2. Capital Budgeting

Let's have one more example. A real estate firm is considering five possible development projects. The estimated long-run profit (net present value) that each project would generate and the amount of investment required to undertake the project are, in units of millions of dollars, presented in Table 5-2. The company has raised $20 million of investment capital for these projects. The resultant formulation is

$$\max \quad z = x_1 + 1.8x_2 + 1.6x_3 + 0.8x_4 + 1.4x_5$$

(5-6) s.t. $$6x_1 + 12x_2 + 10x_3 + 4x_4 + 8x_5 \leq 20$$

$$x_1 , \quad x_2 , \quad x_3 , \quad x_4 , \quad x_5 = 0 \text{ or } 1$$

Excel formulation is presented in Figure 5-4.

Table 5-2: NPV for the Projects and Capital Required

Project	Estimated profit (million dollars)	Capital required (million dollars)
x_1	1.0	6
x_2	1.8	12
x_3	1.6	10
x_4	0.8	4
x_5	1.4	8

Figure 5-4. Excel Formulation for Real Estate Project

5.4. Logical Conditions

IP allows one to depict logical conditions. Some examples are:

- Conditional use: A warehouse can only be used **if** constructed
- Complementary products: **If** any of product A is produced, then a minimum quantity of B must be produced
- Complementary equipment: **If** a particular class of equipment is purchased then only complementary equipment can be acquired
- Sequencing: Operation A must be entirely finished before operation B can begin

is made, the variable takes the value 1. If not, the variable takes the value 0. If the indicator variable takes the value 0, then other activities must also be zero. If the value is 1, then these variables can be non-zero and will be limited by other constraints in the model. An indicator variable is imposed using a constraint as following (large M technique)

$$(5\text{-}7) \quad \sum_i x_i - My \leq 0$$

where M is a large positive number, x_i is a group of continuous variables in the model, and y is an indicator variable (0 or 1). The indicator variable y indicates whether or not any of x's are non-zero with y = 1, zero otherwise. Note that M must be as large as any reasonable value for the sum of the x's. To understand equation (5-7) say we have only two decision variables x_1 and x_2; thus equation (5-7) is $x_1 + x_2 - 10000y \leq 0$. When x_1 and/or x_2 are nonzero, the only way to satisfy the constraint is that y = 1. When both x_1 and x_2 are zero, y may take a value of 0.

Indicator variables, y, may be used in many ways. For example, consider a problem involving two mutually exclusive products, x_1 and x_2. Such a problem may be formulated using the constraints

$$
(5\text{-}8) \quad
\begin{array}{ccccccccc}
x_1 & & & - & My_1 & & & \leq & 0 \\
& & x_2 & & & - & My_2 & \leq & 0 \\
& & & & y_1 & + & y_2 & \leq & 1 \\
x_1 & , & x_2 & & & & & \geq & 0 \\
& & & & y_1 & , & y_2 & = & 0 \text{ or } 1
\end{array}
$$

Here, y_1 indicates whether or not x_1 is produced, while y_2 indicates whether or not x_2 is produced. The third constraint, $y_1 + y_2 \leq 1$, in conjunction with the zero-one restriction on y_1 and y_2, imposes mutual exclusivity. Thus, when $y_1 = 1$ then x_1 can be produced but x_2 cannot. Similarly, when $y_2 = 1$ then x_1 must be zero while $0 \leq x_2 \leq M$. Consequently, either x_1 or x_2 can be produced, but not both. Note that both x_1 and x_2 can be zero meaning that $y_1 = 0$ and $y_2 = 0$. If the third constraint becomes $y_1 + y_2 = 1$, then one of x_1 and x_2 must be produced.

Equation (5-8) may be extended to pair of constraints which is allowed to be active (either-or-active constraints). Say there are two decision variables, then

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & & & - & My_1 & & & \leq & b_1 \\
& & & a_{21}x_1 & + & a_{22}x_2 & & - & My_2 & \leq & b_2 \\
& & & & & & y_1 & + & y_2 & = & 1 \\
& & & & & & y_1 & , & y_2 & = & 0 \text{ or } 1
\end{array}
$$

(5-9)

If $y_1 = 0, y_2 = 1$, the first constraint, $a_{11}x_1 + a_{12}x_2 \leq b_1$, is active. If $y_1 = 1, y_2 = 0$, the second constraint $a_{21}x_1 + a_{22}x_2 \leq b_2$, is active.

5.4.1. AAA Company

The R&D division of the AAA company has developed three possible new products. Each of these products can be produced in **either** of two plants, Plant A or Plant B. However, to avoid undue diversification of the company's product line, management has imposed the following restrictions

(1) From the three possible new products, <u>at most two</u> should be chosen to be produced
(2) Just <u>one of the two plants should be chosen</u> to be the sole producer of the new products

Data for AAA company to construct a LP model is given in Table 5-3. Here are steps to build the LP model. Note that sales potential is an upper limit of each decision variable.

Table 5-3: Data for AAA Company

	Labor used			Labor
	Product 1	Product 2	Product 3	Available
Plant A	3	4	2	30 hours
Plant B	4	6	2	40 hours
Unit profit	5	7	3	in dollars
Sales potential	7	5	9	units

Step 1: *this is a standard product mix problem*: The objective is to choose the products, x_1, x_2, and x_3, the plant and production rates of the chosen products so as to maximize total profit

$$
\begin{array}{llllllllll}
\max & z= & 5x_1 & + & 7x_2 & + & 3x_3 \\
\text{s.t.} & & 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 30 & \text{Labor at plant 1} \\
& & 4x_1 & + & 6x_2 & + & 2x_3 & \leq & 40 & \text{Labor at plant 2} \\
& & x_1 & & & & & \leq & 7 & \text{Sales potential upper limits} \\
& & & & x_2 & & & \leq & 5 \\
& & & & & & x_3 & \leq & 9 \\
& & x_1 & , & x_2 & , & x_3 & \geq & 0 & \text{Non-negativity}
\end{array}
$$

(5-10)

Step 2: *restriction 1: at most two should be chosen to be produced*: The number of strictly positive decision variables must be ≤ 2. If the decision variables were binary variables, then constraint would be $x_1 + x_2 + x_3 \leq 2$. However, here, decision variables are continuous. We need to introduce of auxiliary binary variables. Introduce three auxiliary binary variables (y_1, y_2 and y_3) with the interpretation; $y_j = 1$ if $x_j > 0$ (produce product j) or $y_j = 0$ if $x_j = 0$ (not produce product j). To enforce this interpretation in the model with the help of M (a large positive number), we add the following constraints

$$
\begin{array}{rcl}
x_1 \quad - \quad My_1 & \leq & 0 \\
x_2 \quad - \quad My_2 & \leq & 0 \\
(5\text{-}11) \qquad x_3 \quad - \quad My_3 & \leq & 0 \\
y_1 \; + \; y_2 \; + \; y_3 & \leq & 2 \\
y_1 \; , \; y_2 \; , \; y_3 & = & 0 \text{ or } 1, \text{ binary}
\end{array}
$$

Step 3: *restriction 2: just one of the two plants produces*: It is the either-or constraint, that is, either $3x_1 + 4x_2 + 2x_3 \leq 30$ or $4x_1 + 6x_2 + 2x_3 \leq 40$. To deal with this, we introduce another auxiliary binary variable y_4 with the interpretation; $y_4 = 1$ if the firm chooses plant B or $4x_1 + 6x_2 + 2x_3 \leq 40$ must hold or $y_4 = 0$ if the firm chooses plant A or $3x_1 + 4x_2 + 2x_3 \leq 30$ must hold. This interpretation is enforced by adding the following constraints,

$$
\begin{array}{rcl}
3x_1 \; + \; 4x_2 \; + \; 2x_3 \; - \; My_4 & \leq & 30 \\
(5\text{-}12) \quad 4x_1 \; + \; 6x_2 \; + \; 2x_3 \; - \; M(1-y_4) & \leq & 40 \\
y_4 & = & \text{binary}
\end{array}
$$

Consequently, combining equations (5-10), (5-11), and (5-12) generates the complete model (MIP problem) which is

$$
\begin{array}{rl}
\max \quad z= & 5x_1 \; + \; 7x_2 \; + \; 3x_3 \\
\text{s.t.} \quad & x_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \leq \quad 7 \\
& x_2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq \quad 5 \\
& x_3 \qquad\qquad\qquad\qquad\qquad\qquad \leq \quad 9 \\
& x_1 \qquad - \; My_1 \qquad\qquad\qquad\qquad \leq \quad 0 \\
& x_2 \qquad\qquad - \; My_2 \qquad\qquad\quad \leq \quad 0 \\
(5\text{-}13) \quad & x_3 \qquad\qquad\qquad - \; My_3 \qquad \leq \quad 0 \\
& y_1 \; + \; y_2 \; + \; y_3 \qquad\qquad \leq \quad 2 \\
& 3x_1 \; + \; 4x_2 \; + \; 2x_3 \qquad\qquad\qquad - \; My_4 \quad \leq \quad 30 \\
& 4x_1 \; + \; 6x_2 \; + \; 2x_3 \qquad\qquad\qquad + \; M(1-y_4) \quad \leq \quad 40 \\
& x_1 \; , \; x_2 \; , \; x_3 \qquad\qquad\qquad\qquad\qquad\qquad \geq \quad 0 \\
& y_1 \; , \; y_2 \; , \; y_3 \; , \qquad\qquad y_4 \quad = \quad \text{Binary}
\end{array}
$$

Optimal solution is found using the Excel Solver, that is, $x_1 = 5.5$, $x_2 = 0$ and $x_3 = 9$; $y_1 = 1$, $y_2 = 0$, $y_3 = 1$ and $y_4 = 1$.

5.4.2. BBB Manufacturing (Contingent Decision)

The BBB Manufacturing is considering expansion by building a new factory in either city A or city B, or perhaps in both cities. It also is considering building at most one new ware house, but the choice of location is restricted to a city where a new factory is being built. Economists in the company estimated the expected return (net present value) of each of these alternatives and capital required for the respective investments, where the total capital available is $10 million. The objective is to find the feasible combination of alternatives that maximizes the total net present value.

Let F_A is a binary variable for the question "build factory in city A" and F_B in City B. Similarly, W_A is a binary variable for the question "build warehouse in city A" and W_B in City B.

Because the last two decisions represent mutually exclusive alternatives (the company wants at most one new warehouse), we need the constraint $W_A + W_B \leq 1$. The choice of warehouse locations (W_A and W_B) is contingent decisions; contingent on decisions F_A and F_B, respectively. Thus, $W_A = 0$ if $F_A = 0$, or $W_A = 0$ or 1 if $F_A = 1$. This restriction is imposed by adding the constraint $W_A \leq F_A$ or $-F_A + W_A \leq 0$. Excel formulation is presented in Figure 5-5.

Table 5-4: Data for BBB Company

Yes-or-no question	Decision variable	Expected Return	Capital Required
Build factory in city A?	F_A	$9 million	$6 million
Build factory in city B?	F_B	$5 million	$3 million
Build warehouse in city A?	W_A	$6 million	$5 million
Build warehouse in city B?	W_B	$4 million	$2 million

LP formulation is given by equation (5-14):

$$
\begin{array}{lrcrcrcrcll}
\max \quad z = & 9F_A & + & 5F_B & + & 6W_A & + & 4W_B \\
\text{s.t.} & 6F_A & + & 3F_B & + & 5W_A & + & 2W_B & \leq & 10 & \text{Total capital available} \\
& & & & & W_A & + & W_B & \leq & 1 & \underline{\text{Mutually exclusive}} \\
& -F_A & & & & W_A & & & \leq & 0 & \text{Contingent decision} \\
& & - & F_B & & & + & W_B & \leq & 0 & \text{Contingent decision} \\
& F_A & , & F_B & , & W_A & , & 4W_B & & \text{Binary}
\end{array}
$$

(5-14)

	A	B	C	D	E	F	G	H	I
1	**BBB Manufacturing**								
2		Factory	Factory	Warehouse	Warehouse				
3		in city A	in city B	in city A	in city B	**MAX**			
4		**FA**	**FB**	**WA**	**WB**	**NPV**			
5		1	1	0	0	14			
6		9	5	6	4				
7								RHS	
8	Capital required	6	3	5	2	9	<=	10	Capital req
9	Exclusive decision			1	1	0	<=	1	Exclusive
10	Contigent decision	-1		1		-1	<=	0	Contigent
11	Contigent decision		-1		1	-1	<=	0	Contigent
12									
13		FA, FB, WA, WB are binary (0 or 1)							
14									
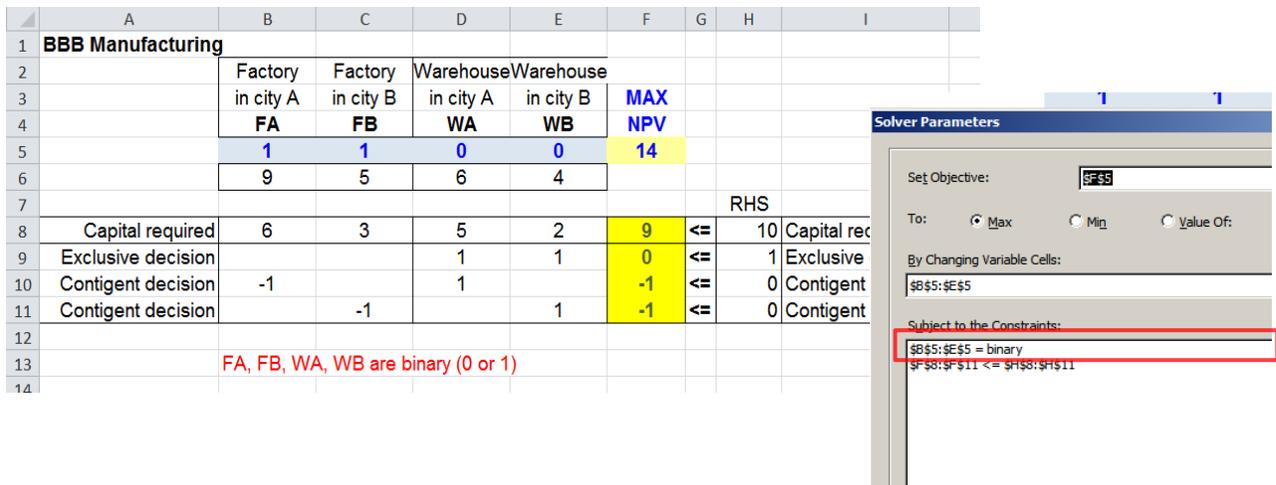
Figure 5-5. Excel Formulation for BBB Company

5.5.   Sensitivity Analysis and Integer Programming

Given the concentration of the book on problem formulation and solution interpretation, we do not consider sensitivity analysis for the integer programming (IP) models. It is because duality is not a well-defined in IP models. Most LP (and nonlinear programming model in the next chapter) duality relationships and interpretations are derived from the calculus; however calculus cannot be applied to the discontinuous integer programming feasible solution region. In general, dual variables are not defined for IP problems. When the IP model is solved using the Excel Solver, Solver Results window doesn't provide Sensitivity Analysis, just Answer option in Reports box in Solver Results window as shown in Figure 5-6.
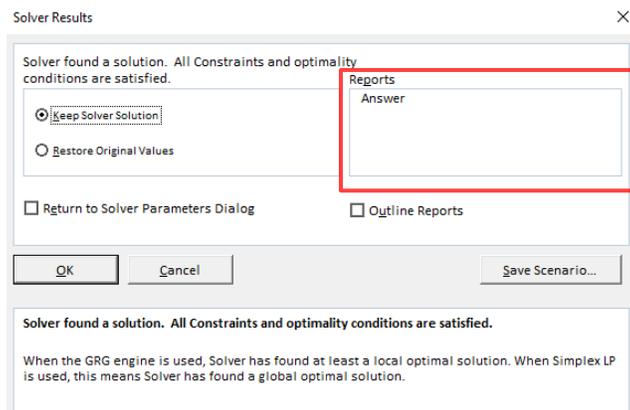
Figure 5-6. Excel Solver Results for Integer Programming Model

5.6. Solution Approach to Integer Programming

IP problems are notoriously difficult to solve. They can be solved by several very different algorithms. We will briefly discuss algorithms, attempting to expose readers to their characteristics.

5.6.1. Rounding

Rounding is the most naive approach to IP problem solution. The rounding approach involves the solution of the problem as a LP problem followed by an attempt to round the solution to an integer one by: a) dropping all the fractional parts; or b) searching out satisfactory solutions wherein the variable values are adjusted to nearby larger or smaller integer values. In general, rounding is often practical, but it should be used with care because it may not provide the correct solution as shown in Section 5.2.

5.6.2. Branch and Bound

The second solution approach developed was the branch and bound algorithm. The algorithm starts with a LP solution (without integer constraint) and then impose constraints to force the LP solution to become an integer solution. Let's consider the following very simple LP problem:

$$
\begin{array}{llrcrcll}
& \max \quad z = & 3x_1 & + & 2x_2 & & & \\
(5\text{-}15) & \text{s.t.} & 2x_1 & + & 2x_2 & \leq & 9 & (1) \\
& & x_1 & , & x_2 & \geq & 0 & \text{Non-negativity}
\end{array}
$$

Given the <u>non-integer</u> optimal solution for (5-15) is $(x_1, x_2) = (4.5, 0.0)$ and $z = 13.50$, the branch and bound algorithm would impose constraints requiring x_1 to be at or below the adjacent integer values around 4.5, that is, $x_1 \leq 4$ and $x_1 \geq 5$. This leads to two disjoint problems such that,

$$
\begin{array}{llrcrcll}
& \max \quad z = & 3x_1 & + & 2x_2 & & & \\
& \text{s.t.} & 2x_1 & + & 2x_2 & \leq & 9 & (1) \\
(5\text{-}16) & & x_1 & & & \leq & 4 & (2) \\
& & x_1 & , & x_2 & \geq & 0 & \text{Non-negativity}
\end{array}
$$

And

$$
\begin{array}{llrcrcll}
& \max \quad z = & 3x_1 & + & 2x_2 & & & \\
& \text{s.t.} & 2x_1 & + & 2x_2 & \leq & 9 & (1) \\
(5\text{-}17) & & x_1 & & & \geq & 5 & (2) \\
& & x_1 & , & x_2 & \geq & 0 & \text{Non-negativity}
\end{array}
$$

The branch and bound solution procedure generates two problems (branches) after each LP solution as in equation (5-16) and (5-17). Each problem excludes the unwanted non-integer solution, forming an increasingly more tightly constrained LP problems. The optimal solution to the model in (5-16) is $(x_1, x_2) = (4, 0.5)$ and $z = 13.00$, while the model in (5-17) is infeasible (no solution). Because x_2 is not

integer, the branch and bound imposes constraints (again) requiring x_2 to be at or below the adjacent integer values around 0.5, that is, $x_2 \leq 0$ and $x_2 \geq 1$. This leads to another set of two disjoint problems.

In doing this, when one solves a particular problem, one may find an integer solution. However, one cannot be sure it is optimal until <u>all</u> problems have been examined. Maximization problems will exhibit declining objective function values whenever additional constraints are added. Consequently, given a feasible integer solution has been found, then any solution, integer or not, with a smaller objective function value cannot be optimal, nor can further branching on any problem below it yield a better solution than the incumbent since the objective function will only decline. Thus, the best integer solution found at any stage of the algorithm provides a bound limiting the problems (branches) to be searched. The bound is continually updated as better integer solutions are found. The branch and bound ends if the last branch is infeasible or integer solution.

For the example in equation (5-15), there are 21 sub-LP problems to solve and the optimal solution is $(x_1^*, x_2^*) = (4, 0)$ and $z^* = 12$.

6. NONLINEAR PROGRAMMING

Key points:

LP linear function assumption is relaxed; objective function and/or constraints could be nonlinear. In this case, optimal solution doesn't occur on the corner point feasible (CPF) solution and thus different solution search algorithm is needed for example Newton-Raphson method.

In Excel Solver, select "GRG Nonlinear" in Select a Solving Method window after entering nonlinear formulas in the corresponding cells.

6.1. Introduction

A key assumption of linear programming is that all its functions (objective function and constraints) are linear. Although this assumption essentially holds for numerous practical problems, it frequently does not hold. Many economists have found that some degree of nonlinearity is the rule and not the exception in economic problem. We now turn our attention to continuous, certain, nonlinear optimization problems.

Nonlinearity can arise in various ways. In the production problem in LP, the per-unit profit of each product was assumed to be constant. But it can be a decreasing function of the output level, either because a larger output tends to depress the market price or because increased production tends to raise the (average) variable cost of the product. If so, the LP objective function $z = c_1 x_1 + \cdots + c_n x_n$ must be replaced by a nonlinear version, such as $z = c_1(x_1)x_1 + \cdots + c_n(x_n)x_n$, where $c_j(x_j)$ denotes a decreasing function of the variable x_j.

Let's consider a simple example:

$$
\begin{array}{lllllllll}
\max & z = & 126x_1 & - & 9x_1{}^2 & + & 182x_2 & - & 13x_2{}^2 \\
\text{s.t.} & & x_1 & & & & & \leq & 4 \\
\text{(6-1)} & & & & 9x_1{}^2 & + & 5x_2{}^2 & \leq & 216 \\
& & x_1 & & & x_2 & & \geq & 0
\end{array}
$$

There are two decision variables x_1 and x_2 with nonlinear objective function and constraint. Graphical illustration of this NLP is presented in Figure 6-1 and Excel formulation in Figure 6-2. Note that optimal solution doesn't occur on the corner point feasible (CPF) solution.

$z^* = 895.14$
$x_1 = 3.284$ and $x_2 = 4.878$

Note that optimal solution doesn't occur on the corner point feasible (CPF) solution
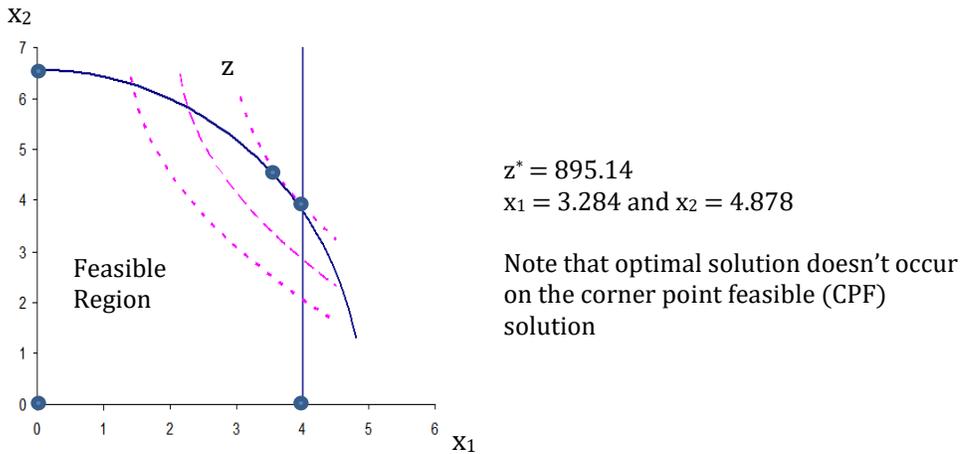
Figure 6-1. Graphical Solution of Nonlinear Programming
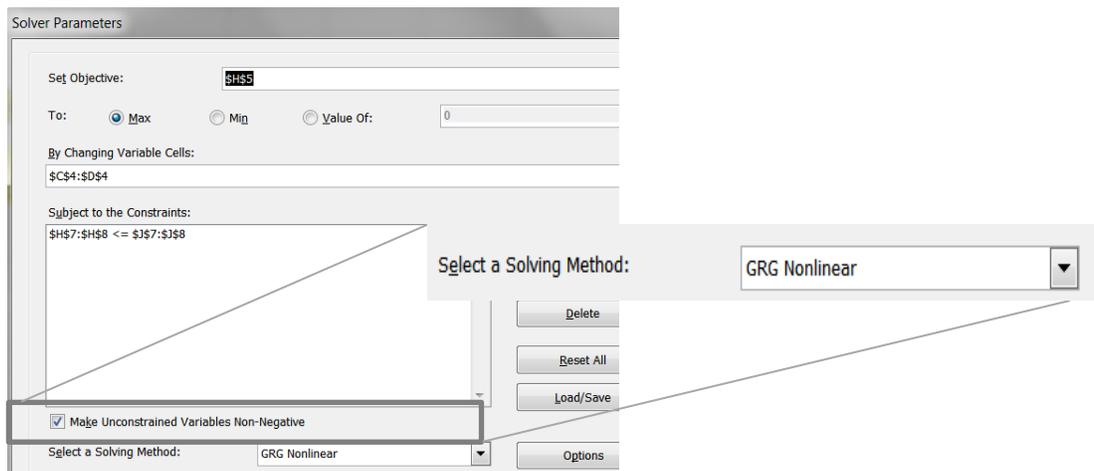

Figure 6-2. Excel Formulation for Nonlinear Programming

6.2. Solving Nonlinear Programming Model

Because the optimal solution doesn't occurs on the CPF solution, the Simplex doesn't work. Instead, searching (optimal solution) procedure is used. Basic idea is

1) test current trial solution if derivative is negative or positive,

2) compute next trial solution (iteration),

3) test again,

4) compute next trial solution,

5) test again and stop if stopping rule is satisfied (convergence criteria)

Popular methods are 1) gradient method, 2) Newton's method, and 3) Frank-Wolfe algorithm.

In Excel, just select **GRG Nonlinear** in Select a Solving Method window as shown in Figure 6-2 and run the model.

6.3. Quadratic Programming

Quadratic programming problems have linear constraints but the objective function must be *quadratic*, meaning that it has the square of a variable, x_j^2, or the product of two variables, $x_i x_j$ ($i \neq j$) terms. For example,

$$
\begin{array}{rlcccccccc}
\max & z = & 15x_1 & + & 30x_2 & + & 4x_1x_2 & - & 2x_1^2 & - & 4x_2^2 \\
(6\text{-}2) \quad \text{s.t.} & & x_1 & + & x_2 & & & & & \leq & 30 \\
& & x_1 & , & x_2 & & & & & \geq & 0
\end{array}
$$

We will have the quadratic programming models in the next chapter, Price Endogenous Modeling.

7. PRICE ENDOGENOUS MODELING

Key points:

A common economic application of nonlinear programming involves price endogenous models where prices are endogenously determined in the model; thus it involves modeling an industry or sector.

In Excel Solver, select "GRG Nonlinear" to solve the price endogenous model.

7.1. Introduction

A common economic application of nonlinear programming involves price endogenous models. In the standard LP model, input and output prices or quantities are assumed fixed and exogenous. Price endogenous models are used in situations where this assumption is felt to be untenable. Such problems can involve modeling an industry or sector such that the level of output or purchases of inputs is expected to influence equilibrium prices.

Let an inverse demand equation be defined as $P_d = a_d - b_d Q_d$, where P_d is price of the product, $a_d > 0$ is the (demand) intercept, $b_d > 0$ is the (demand) slope, and Q_d is the quantity demanded. Similarly, suppose we have an inverse supply equation, $P_s = a_s + b_s Q_s$, where the terms are defined analogously. An equilibrium solution would have price and quantity equated and would occur at the simultaneous solution of the equations

(7-1)
$$P_d = P_s \text{ or } a_d - b_d Q_d = a_s + b_s Q_s \text{ \& } Q_d = Q_s = Q^*$$
$$\rightarrow Q^* = \frac{a_d - a_s}{b_d + b_s} \text{ and } P^* = a_d - b_d Q^* = a_s + b_s Q^*$$

The equilibrium price and quantity is P^* and Q^* as in Figure 7-1.

Consumer surplus (CS) is defined as the difference between the maximum price a consumer is willing to pay and the actual price they do pay, P^*. If a consumer would be willing to pay more than the current asking price, then they are getting more benefit from the purchased product than they initially paid. Graphically it is the area under the demand and above the equilibrium price. Producer surplus (PS) is a measure of producer welfare. It is measured as the difference between what producers are willing and able to supply a good for and the price they actually receive. Graphically it is the area above the supply curve and below the market price. The sum of CS and PS is considered as the social welfare (Figure 7-2). Mathematically,

(7-2)
$$SW = \int_0^{Q^*} P_d \, dQ - \int_0^{Q^*} P_s \, dQ$$
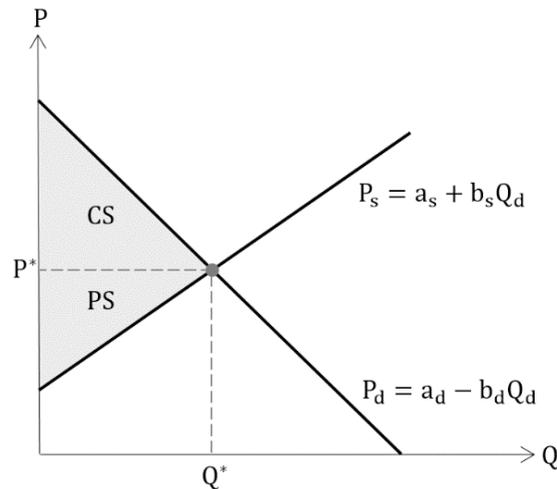
Figure 7-1. Market Equilibrium; Consumer and Producer Surplus

Assuming linear demand and supply, equation (7-2) becomes

(7-3)
$$SW = \int_0^{Q^*} (a_d - b_d Q_d)\, dQ - \int_0^{Q^*} (a_s + b_s Q_s)\, dQ$$

$$= \left(a_d Q_d - 0.5 b_d Q_d^2\right) - \left(a_s Q_s + 0.5 b_s Q_s^2\right)$$

And, in turn, we set up the model to maximize SW with a demand-supply balance constraint such that

$$
\begin{array}{llllll}
\max & SW = & a_d Q_d & - & 0.5 b_d Q_d^2 & - & a_s Q_s & - & 0.5 b_s Q_s^2 \\
\text{s.t.} & & Q_d & & & - & Q_s & & \leq & 0 & \text{Demand-supply balance} \\
& & Q_d & , & & & Q_s & & \geq & 0
\end{array}
$$

(7-4)

This is a nonlinear model (quadratic programming model). Note that P^* is the shadow price of the first constraint.

Suppose we have

(7-5) $P_d = 60 - 3Q_d$ and $P_s = 10 + 2Q_s$

The above example is a simple case where we have a single supply and single demand curve. Clearly, no one would solve this problem using nonlinear programming, as it could be easily solved by hand. Let $P_d = P_s$ or $60 - 3Q_d = 10 + 2Q_s$ and $Q_d = Q_s = Q^*$. So, $Q^* = 10$ and $P^* = 30$. The price endogenous model is

$$\max \quad SW = \quad 60Q_d \quad - \quad 1.5Q_d^2 \quad - \quad 10Q_s \quad - \quad Q_s^2$$

$$(7\text{-}6) \quad \text{s.t.} \qquad\qquad Q_d \qquad\qquad - \qquad Q_s \qquad \leq \quad 0$$

$$\qquad\qquad\qquad\qquad Q_d \quad , \qquad\qquad\qquad Q_s \qquad \geq \quad 0$$

Figure (7-3) presents the Excel formulation and the corresponding sensitivity report where we can find the equilibrium price (shadow price, "Lagrange multiplier", of the demand-supply balance constraint). The SW is 250.

	A	B	C	D	E	F	G	H	I
1	**Price Endogenous Model**								
2									
3		Demand paramters			Supply parameters				
4		ad	60		as	10.0			
5		bd	-3		bs	2			
6									
7				Qd	Qs		MAX		
8		Changing cells		**10.00**	**10.00**		**250.00**		
9							Used		RHS
10		Constraint		1	-1		**0.00**	<=	0
11									

`=(C5*D9+0.5*C6*D9^2)-(F5*E9+0.5*F6*E9^2)`

`=SUMPRODUCT(D9:E9,D11:E11)`

6	Variable Cells			
7			**Final**	**Reduced**
8	**Cell**	**Name**	**Value**	**Gradient**
9	$D$8	Changing cells Qd	10	0
10	$E$8	Changing cells Qs	10	0
11				
12	Constraints			
13			**Final**	**Lagrange**
14	**Cell**	**Name**	**Value**	**Multiplier**
15	$G$10	Constraint Used	0	30.00

Figure 7-2. Price Endogenous Model and Sensitivity Report

Note that the formulation was originally motivated by Enke (1951)[2] and Samuelson (1952)[3]. Later it was fully developed by Takayama and Judge (1971)[4].

---

[2] Enke, S. (1951) "Equilibrium among Spatially Separated Markets: Solution by Electric Analogue." *Econometrica* 19: 40-47.

[3] Samuelson, P.A. (1952) "Spatial Price Equilibrium and Linear Programming." *American Economic Review* 42: 283-303.

[4] Takayama, T. and Judge, G. (1971) *Spatial and Temporal Price and Allocation Models*. Amsterdam, North Holland.

7.2. Spatial Equilibrium

A common price endogenous model application involves the spatial equilibrium problem. This problem is an extension of the transportation problem relaxing the assumption of fixed supply and demand; that is, supply constraints, s_i, and demand requirements, d_j, in equation (4-6) in Chapter 4 are now endogenous. The problem is motivated as follows. Production and/or consumption usually occurs in spatially separated regions, each of which have supply and demand relations. In a solution, if the regional prices differ by more than the interregional cost of transporting goods, then trade will occur and the price difference will be driven down to the transport cost. Modeling of this situation addresses the questions of who will produce and consume what quantities and what level of trade will occur. Takayama and Judge (1971) developed the spatial equilibrium model to deal with such situations.

Suppose that inverse corn supply function in the US is $P_{s,US} = 25 + Q_{s,US}$ and no supply in Japan. Inverse corn demand functions are $P_{d,US} = 150 - Q_{d,US}$ for the US and $P_{d,JP} = 160 - Q_{d,JP}$, respectively. If there is no trade, the US market is cleared where $P_{s,US} = P_{d,US}$; $P_{US}^* = 87.5$ and $Q_{US}^* = 62.5$ (Figure 7-3). Suppose that transport between the US and Japan costs 4. US producers will export corn to Japan when the (international) price is higher than 91.5 = 87.5 + 4. Thus the inverse corn supply in international market (from the U.S.) is $P_d = 91.5 + 0.5Q_s$ and the international market will be cleared when $91.5 + 0.5Q^* = 160 - Q^*$ or $P^* = 114.3$ and $Q^* = 45.7$ (international market in Figure 7-4). Note that the supply slope in the international market is 0.5 because producers will supply half of corn to the U.S. and half of corn to the international market. Market price in Japan is 114.3 and market price in the US is 110.3 (4 less); US producers produce 85.3 units of corn and export 45.7 units.
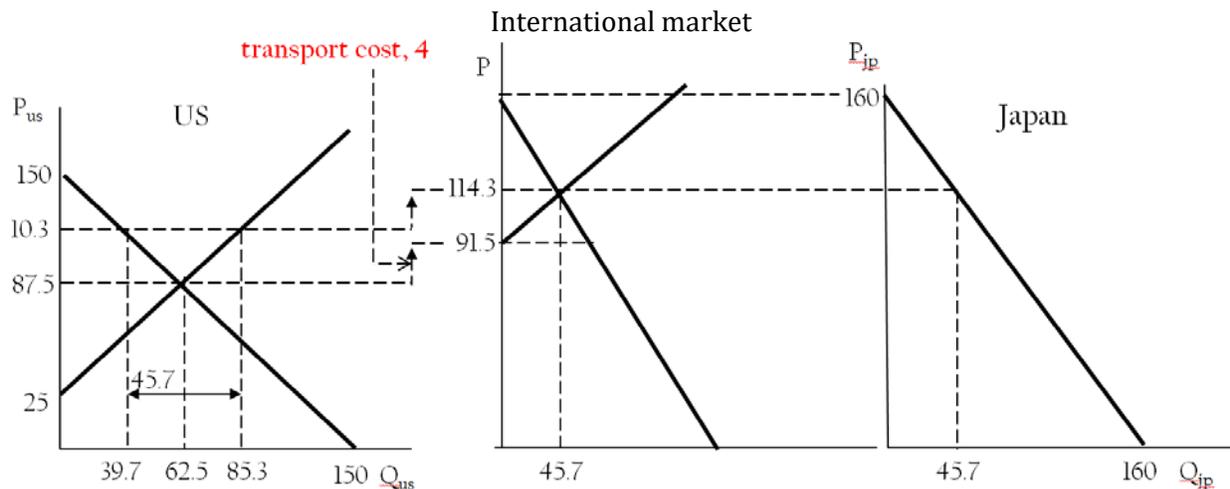
Figure 7-3. Spatial Equilibrium Graphical Solution

Social welfare (SW) is this example is the sum of consumer surpluses in the US and Japan and producer surpluses in the US and Japan (note that producer surplus in Japan is zero because there is no supply). The net welfare (NW) is the difference of SW and the transportation cost, or

$$
\text{NW} = \left[\int_0^{Q_{US}^*} P_{d,US}\, dQ_{US} - \int_0^{Q_{US}^*} P_{s,US}\, dQ_{US} + \int_0^{Q_{JP}^*} P_{d,JP}\, dQ_{JP} - \int_0^{Q_{JP}^*} P_{s,JP}\, dQ_{JP} \right]
$$

(7-7)

$$
- c_{US,US} T_{US,US} - c_{US,JP}\, T_{US,JP}
$$

where $c_{US,JP}$ is the unit transport cost (which is 4) and $T_{US,JP}$ is the amount of transported to Japan from the US (export). In turn, as we did in the previous section, we may form an optimization problem with the NW expression as the objective function plus the constraints from the transportation model. Let T_{ij} (i, j = US, JP) is the shipment form i to j, and thus T_{ii} is the domestic or internal shipment (domestic supply assuming zero cost). The constraints involve a demand balance requiring that incoming shipments to a region be greater than or equal to regional demand, that is, $Q_{d,i} \leq T_{i,i} + T_{i,j}$, and a supply balance requiring that outgoing shipments do not exceed regional supply, $Q_{s,i} \geq T_{i,i} + T_{i,j}$. The resultant problem becomes

$$
\begin{array}{lllllll}
\max \quad \text{NW} = & 150 Q_{d,US} & - & 0.5 Q_{d,US}^2 & - & 25 Q_{s,US} & - & 0.5 Q_{s,US}^2 & & & \text{US SW} \\
& + \quad 160 Q_{d,JP} & - & 0.5 Q_{d,JP}^2 & & & & & & & \text{JP SW} \\
& - \quad 0 T_{US,US} & - & 4 T_{US,JP} & & & & & & & \text{transportation} \\
\text{s.t.} & Q_{d,US} & - & T_{US,US} & - & T_{JP,US} & & & \leq & 0 & \text{US DMD} \\
& Q_{d,JP} & - & T_{US,JP} & - & T_{JP,JP} & & & \leq & 0 & \text{JP DMD} \\
& -Q_{s,US} & + & T_{US,US} & + & T_{JP,US} & & & \leq & 0 & \text{US SPL} \\
& Q_{d,i} & , & Q_{s,i} & , & T_{i,j} & & & \geq & 0 &
\end{array}
$$

(7-8)

Based on the discussion above, we derive the general spatial equilibrium model. Suppose that in region i the demand for the good of interest is given by $P_{di} = f_i(Q_{di})$ where P_{di} is the demand price in region i while Q_{di} is the quantity demanded. Similarly in region i the supply for the good is given by $P_{si} = g_i(Q_{si})$ where P_{si} is the supply price in region i and Q_{si} is the quantity supplied. A social welfare (SW) function for each region can be defined as the area between the supply and demand curves (sum of CS and PS):

(7-9)
$$
SW_i(Q^*) = \int_0^{Q_i^*} P_{d,i}\, dQ_i - \int_0^{Q_i^*} P_{s,i}\, dQ_i
$$

The net welfare (NW) function across all regions is the sum of the welfare functions in each region less total transport costs. Suppose T_{ij} represents the amount of good shipped from i to j at cost c_{ij}. Then the NW is

$$(7\text{-}10) \quad NW(Q^*) = \sum_i^N \left[\int_0^{Q_i^*} P_{d,i} \, dQ_i - \int_0^{Q_i^*} P_{s,i} \, dQ_i \right] - \sum_i^N \sum_j^N c_{ij} T_{ij}$$

A demand balance requiring that incoming shipments to a region be greater than or equal to regional demand:

$$(7\text{-}11) \quad Q_{di} \leq \sum_j T_{ji} \quad \text{for all i}$$

and a supply balance requiring that outgoing shipments do not exceed regional supply:

$$(7\text{-}12) \quad Q_{si} \geq \sum_j T_{ij} \quad \text{for all i}$$

All together, the resultant problem becomes

$$(7\text{-}13) \quad \begin{aligned} \max \quad NW = &\sum_i^N \left[\int_0^{Q_i^*} P_{d,i} \, dQ_i - \int_0^{Q_i^*} P_{s,i} \, dQ_i \right] - \sum_i^N \sum_j^N c_{ij} T_{ij} \\ \text{s.t.} \quad\quad\quad\quad & Q_{di} - \sum_j T_{ji} && \leq \quad 0 \quad \text{For all i} \\ & -Q_{si} + \sum_j T_{ij} && \leq \quad 0 \quad \text{For all i} \\ & Q_{di}, \quad Q_{si}, \quad T_{ij} && \geq \quad 0 \end{aligned}$$

Note that a shadow price (Lagrange multiplier from Excel Solver Sensitivity Report) for the first constraint equals the demand price and a shadow price for the second constraint equals the supply price. Transportation that the demand price in a region must be less than the supply prices in all other regions plus transport cost. Also, T_{ii} represents the quantity produced in region i and consumed in region i; If region i fills some of its own demand, that is, $T_{ii} > 0$, then supply and demand prices in region i are equal. If region i exports to region j, that is, $T_{ij} > 0$, then the demand price in region j equals the supply price in region i plus transport cost (See Figure 7-3, where $P_{d,J} = P_{s,U} + c_{U,J} \to 114.3 = 110.3 + 4$). If region i doesn't export to region j, that is, $T_{ij} = 0$, then generally $P_{dj} < P_{si} + c_{ij}$; trade is not desirable since the price difference won't support the transport cost.

Example:

Suppose we have three entities (US (U), Europe (E), Japan (J)) trading a single homogeneous commodity. Supply curves are

$$P_{s,U} = 25 + Q_{s,U}$$
$$(7\text{-}14) \quad P_{s,E} = 35 + Q_{s,E}$$
$$P_{s,J} = 100 + Q_{s,J}$$

while the demand curves are

$$P_{d,U} = 150 - Q_{d,U}$$
$$(7\text{-}15) \quad P_{d,E} = 155 - Q_{d,E}$$
$$P_{d,J} = 160 - Q_{d,J}$$

and internal transport is zero, i.e., $c_{ii} = 0$. Also suppose transport between the US and Europe costs 3 in either direction, while it costs 4 between the US and Japan and 5 between Europe and Japan. The formulation of this problem based on equation (7-13) is

$$
\begin{array}{lllllll}
\max \quad NW = & 150Q_{d,U} & - \ 0.5Q_{d,U}^2 & - \ 25Q_{s,U} & - \ 0.5Q_{s,U}^2 & & \text{US SW} \\
& + \ 155Q_{d,E} & - \ 0.5Q_{d,E}^2 & - \ 35Q_{s,E} & - \ 0.5Q_{s,E}^2 & & \text{EU SW} \\
& + \ 160Q_{d,J} & - \ 0.5Q_{d,J}^2 & - \ 100Q_{s,J} & - \ 0.5Q_{s,J}^2 & & \text{JP SW} \\
& - \ 0T_{U,U} & - \ 3T_{U,E} & - \ 4T_{U,J} & & & \text{Trnsprt from US} \\
& - \ 3T_{E,U} & - \ 0T_{E,E} & - \ 5T_{E,J} & & & \text{Trnsprt from EU} \\
& - \ 4T_{J,U} & - \ 5T_{J,E} & - \ 0T_{J,J} & & & \text{Trnsprt from JP} \\
\end{array}
$$

$$
(7\text{-}16) \quad \text{s.t.}
\begin{array}{lllllll}
Q_{d,U} & - \ T_{U,U} & - \ T_{E,U} & - \ T_{J,U} & \le & 0 & \text{US DMD} \\
Q_{d,E} & - \ T_{U,E} & - \ T_{E,E} & - \ T_{J,E} & \le & 0 & \text{EU DMD} \\
Q_{d,J} & - \ T_{U,J} & - \ T_{E,J} & - \ T_{J,J} & \le & 0 & \text{JP DMD} \\
-Q_{s,U} & + \ T_{U,U} & + \ T_{U,E} & + \ T_{U,J} & \le & 0 & \text{US SPL} \\
-Q_{s,E} & + \ T_{E,U} & + \ T_{E,E} & + \ T_{E,J} & \le & 0 & \text{EU SPL} \\
-Q_{s,J} & + \ T_{J,U} & + \ T_{J,E} & + \ T_{J,J} & \le & 0 & \text{JP SPL} \\
Q_{d,i} & , \quad Q_{s,i} & , \quad T_{i,j} & & \ge & 0 & \\
\end{array}
$$

Spatial Equilibrium Model

	Parameters	Demand			Supply			
		US	EU	JP	US	EU	JP	
intercept (a)		150	155	160	25	35	100	intercept
slope (b)		-1	-1	-1	1	1	1	slope

Transport cost

	To	US	EU	JP
From	US	0	3	4
	EU	3	0	5
	JP	4	5	0

Decision variables - Changing Cells

	Q DMD US	Q DMD EU	Q DMD JP	Q SUPPLY US	Q SUPPLY EU	Q SUPPLY JP	T US->US	T US->EU	T US->JP	T EU->US	T EU->EU	T EU->JP	T JP->US	T JP->EU	T JP->JP
changing cells	46.83	52.83	52.83	78.17	67.17	7.17	46.83	0.00	31.33	0.00	52.83	14.33	0.00	0.00	7.17

Intermediate Calculations

	DEMAND			SUPPLY			TRANSPORT								
	US	EU	JP	US	EU	JP									
a*Q(t)+0.5*b*Q(t)^2	5928	6793	7058	-5009	-4607	-742	0	0	-125	0	0	-72	0	0	0
	=C4*C17+0.5*C5*C17^2			=-(F4*F17+0.5*F5*F17^2			=-C9*I17								

MAX

SUM of int. calculations **9224.42** <<<<<<<< objective function =SUM(C21:Q21)   '=SUMPRODUCT($C$17:$Q$17,C27:Q27)

Constraints

		C	F	I	J	K	L	M	N	O	P	Q	USED		RHS
Constraints	Q DMD US	1		-1			-1			-1			0.00	<=	0
	Q DMD EU	1			-1			-1			-1		0.00	<=	0
	Q DEM JP	1				-1			-1			-1	0.00	<=	0
	Q SPL US		-1	1	1	1							0.00	<=	0
	Q SPL EU		-1				1	1	1				0.00	<=	0
	Q SPL JP		-1							1	1	1	0.00	<=	0

Price Calculation

	P US	P EU	P JP		P US
	103.17	102.17	107.17		=C4+C5*C17

Social Welfare Calculation

	US	EU	JP	
CS	1096.68	1395.68	1395.68	=0.5*((C4-C38)*C17)
PS	3055.01	2255.68	25.68	=0.5*((C38-F4)*F17)
SW	4151.69	3651.36	1421.36	=C42+C43

Figure 7-4. Spatial Equilibrium Excel Formulation

Figure 7-4 is the Excel formulation with the solution to the problem. This solution indicates consumption of 47 units in the U.S., and 53 units in both Europe and Japan, while 78 units are supplied (produced) in the US, 67 in Europe, and 7 units in Japan. The U.S. and Europe both get all of their consumption quantities from domestic production while the U.S. exports 31 units to Japan and Europe exports 14 units. The equilibrium prices appear in the row 40 using the parameters in demand equations; the price in the U.S. is 103.17 while the European price is 102.17. Note the Japanese price is 107.17 which is higher than the price in the other two regions by the transport cost.

The utility of this model may be demonstrated by performing some slight extensions. Suppose we use the model to examine the costs and effects of trade barriers and their cost. Specifically consider first model solution without any trade (no trade case) adding additional constraints such that $T_{U,E} + T_{U,J} = 0$ (US export is zero), $T_{E,U} + T_{E,J} = 0$ (EU export is zero) and $T_{J,U} + T_{J,E} = 0$ (Japan export is zero) as in Figure 7-5.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
1	Spatial Equilibrium Model																			
2		Parameters	Demand			Supply														
3			US	EU	JP	US	EU	JP												
4		intercept (a)	150	155	160	25	35	100	intercept											
5		slope (b)	-1	-1	-1	1	1	1	slope											
6																				
7			Transport cost																	
8		To	US	EU	JP															
9	From	US	0	3	4															
10		EU	3	0	5															
11		JP	4	5	0															
12																				
13		Decision variables - Changing Cells																		
14			Q	Q	Q	Q	Q	Q	T	T	T	T	T	T	T	T	T			
15			DMD	DMD	DMD	SUPPLY	SUPPLY	SUPPLY	US->US	US->EU	US->JP	EU->US	EU->EU	EU->JP	JP->US	JP->EU	JP->JP			
16			US	EU	JP	US	EU	JP												
17		changing cells	63	60	30	63	60	30	63	0	0	0	60	0	0	0	30			
18																				
19	Intermediate		DEMAND			SUPPLY			TRANSPORT											
20	Calculations		US	EU	JP	US	EU	JP												
21	$a*Q(t)+0.5*b*Q(t)^2$		7422	7500	4350	-3516	-3900	-3450	0	0	0	0	0	0	0	0	0			
22			=C4*C17+0.5*C5*C17^2			=-(F4*F17+0.5*F5*F17^2	=-C9*I17													
23																				
24			MAX																	
25	SUM of int. calculations		8406.25	<<<<<<<< objective function		=SUM(C21:Q21)										'=SUMPRODUCT($C$17:$Q$17,C27:Q27)				
26																		USED		RHS
27	Constraints	Q DMD US	1						-1			-1			-1			0.00	<=	0
28		Q DMD EU		1						-1			-1			-1		0.00	<=	0
29		Q DEM JP			1						-1			-1			-1	0.00	<=	0
30		Q SPL US				-1			1	1	1							0.00	<=	0
31		Q SPL EU					-1					1	1	1				0.00	<=	0
32		Q SPL JP						-1							1	1	1	0.00	<=	0
33		US Export								1	1							0.00	=	0
34		EU Export										1		1				0.00	=	0
35		JP Export													1	1		0.00	=	0
36																				
37																				
38		Price Calculation																		
39			P US	P EU	P JP		P US													
40			87.50	95.00	130.00		=C4+C5*C17														
41																				
42		Social Welfare Calculation																		
43			US	EU	JP															
44		CS	1953.13	1800.00	450.00		=0.5*((C4-C38)*C17)														
45		PS	1953.13	1800.00	450.00		=0.5*((C38-F4)*F17)														
46		SW	3906.25	3600.00	900.00		=C42+C43														

Figure 7-5. Spatial Equilibrium Excel Formulation - No Trade

Note that the expected results occur. Without trade, domestic consumers in the U.S. and Europe receive cheaper prices and consume more, but Japanese consumers should pay much higher price. Simultaneously U.S. and European producers supply less and receive lower prices. Clearly we can see the benefit of the (free) trade in terms of social welfare (compare objective function values, i.e., 9224 vs. 8406 or values of SW in each region) but there might be distributional issue. Under the free trade (Figure 7-4) the U.S. and Europe producers are better-off but not consumers even if it has higher social welfare. All in all, this example illustrates the potential usefulness of the spatial equilibrium, price endogenous structure.

8. PORTFOLIO ANALYSIS

Key points:

Portfolio theory deals with the problem of constructing for a given collection of financial assets an investment with desirable features.

Mean-variance portfolio involves development of an optimal investment strategy

8.1. Introduction

Portfolio theory deals with the problem of constructing for a given collection of financial assets an investment with desirable features. A financial asset is a non-physical asset whose value is derived from a contractual claim such as bank deposit, bonds and stocks. A variety of different asset characteristics can be taken into consideration, such as the amount of value, on average, an asset returns on over a period of time and the riskiness of reaping returns comparable to the average. The financial objectives of the investor and tolerance of risk determine what types of portfolios are to be considered desirable. In this chapter we will discuss a quantitative approach to constructing portfolios.

8.2. Rates of Return of Assets

There are two basic features of an asset. The first is the average return of an asset over a period of time. The second characteristic is how risky it is to obtain similar returns comparable to the average over the investment period. For an asset with value $S(0)$ at time 0 and value $S(T)$ at time T, the rate of return, r, is defined by

$$(8\text{-}1) \quad \tilde{r} = \frac{S(T) - S(0)}{S(0)} \quad \Leftrightarrow \quad S(T) = (1 + \tilde{r})S(0)$$

For example, if $S(0) = \$4$ and after one month $S(1) = \$5$, the rate of return of the asset is 25%. The rate of return of an asset is sometimes referred to as the yield of the asset.

Since the outcome of an investment in an asset has some level of uncertainty, the value S_T is unknown (to the decision maker) exactly at time 0. To model the uncertainty we shall consider the value of the asset at time T as a **random** variable. Correspondingly, the rate of return defined by equation (8-1) is also a random variable. This is why there is tilde on r in equation (8-1) to indicate that rate or return is random. To characterize the asset we shall consider the *average* rate of return defined by

$$(8\text{-}2) \quad \bar{r} = E(\tilde{r})$$

where E( ) denotes the expectation of a random variable and $\bar{r}$ is the expected rate of return. While the expected rate of return is a useful way to characterize an asset and gives us some indication of how large the returns may be, it does not capture the uncertainty in obtaining a comparable return

rate to the average. To quantify how much the rate of return deviates from the expected return and in order to capture the riskiness of the asset, we shall use the _variance_ defined by:

$$(8\text{-}3) \quad \sigma^2 = \text{Var}(\tilde{r}) = E[(\tilde{r} - \bar{r})^2]$$

For a given collection of n assets $\{S_1, S_2, \cdots, S_n\}$, for the ith asset we denote the rate of return by r_i and the variance by σ_i^2. The covariance for the returns between asset i and j is

$$(8\text{-}4) \quad \sigma_{ij} = E[(\tilde{r}_i - \bar{r}_i)(\tilde{r}_j - \bar{r}_j)]$$

Note that $\sigma_{ij} = \sigma_{ji}$ and $\sigma_{ii} = \sigma_i^2$.

To describe the coupling of all n assets we define the covariance matrix by:

$$(8\text{-}5) \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

Note that Σ is a symmetric matrix. See Appendix of the chapter for more information.

8.3.   Portfolio with Two Assets

8.3.1. Mean, Variance and Covariance

Let's begin with a simple portfolio of two assets, S_1 and S_2 (investment in stocks, for example). Suppose that returns from each asset at the month t is $\tilde{r}_{1,t}$ and $\tilde{r}_{2,t}$, where tilde on the variable indicates it is random meaning that a decision maker doesn't know the return when s/he makes a portfolio. Both assets are characterized by their mean or expected value ($\bar{r}_i, i = 1, 2$), variance (σ_i^2) and covariance (σ_{12}).

Suppose that means are $E(\tilde{r}_1) = \bar{r}_1 = 1.49\%$ and $E(\tilde{r}_2) = \bar{r}_2 = 1.00\%$, respectively; variances are $\sigma_1^2 = 77.7\%$ ($\sigma_1 = 8.81\%$, standard deviation) and $\sigma_2^2 = 35.87\%$ ($\sigma_2 = 5.99\%$); the covariance is $\sigma_{12} = 20.95\%$. Note that the sign of the covariance shows the tendency in the relationship between the assets. The magnitude of the covariance is not easy to interpret because it is not normalized and depends on the magnitudes of the variables. The normalized version of the covariance, the correlation coefficient, ρ_{ij}, however, shows by its magnitude the strength of the linear relation;

$$(8\text{-}6) \quad \rho_{12} = \frac{\sigma_{12}}{\sigma_1 \cdot \sigma_2} = \frac{20.95}{\sqrt{77.70} \cdot \sqrt{35.87}} = 0.397$$

The correlation coefficient has a value between +1 and -1, where +1 is perfect positive linear correlation, 0 is no linear correlation, and -1 is perfect negative linear correlation.

The covariance matrix, Σ, is:

$$\text{(8-7)} \quad \mathbf{\Sigma} = \begin{bmatrix} \text{var}(\tilde{r}_1) & \text{cov}(\tilde{r}_1, \tilde{r}_2) \\ \text{cov}(\tilde{r}_1, \tilde{r}_2) & \text{var}(\tilde{r}_2) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} 77.70 & 20.95 \\ 20.95 & 35.87 \end{bmatrix}$$

A portfolio of these two assets is characterized by the value invested in each asset; let v_1 and v_2 be the dollar amount invested in asset 1 and 2, respectively. The total value of the portfolio is $v = v_1 + v_2$; consider a portfolio in which

$$\text{(8-8)} \quad w_1 = \frac{v_1}{v} \text{ and } w_2 = \frac{v_2}{v}$$

where w_1 is the weight on asset 1 and w_2 is the weight on asset 2; hence, by construction, $w_1 + w_2 = 1$. For example, a decision maker (DM) has \$1,000 to invest in assets 1 and 2. If the DM invests \$500 in asset 1 and \$500 in asset 2, then $w_1 = w_2 = \frac{\$500}{\$1000} = 0.5$ (equally weighted portfolio of the two stocks).

The portfolio return, $\tilde{r}_p$, with two assets is a weighted average of the individual returns, that is,

$$\text{(8-9)} \quad \tilde{r}_p = w_1 \tilde{r}_1 + w_2 \tilde{r}_2$$

For example, suppose the DM invests \$600 in asset 1 ($w_1 = 0.6$) and \$400 in asset 2 ($w_2 = 0.4$) for a month. If the "realized" return is 2% on asset 1 and 1% on asset over the month, then the return on the portfolio is $\tilde{r}_p = 0.6 \cdot 2\% + 0.4 \cdot 1\% = 1.6\%$.

Expected return on a portfolio with two assets[5] is

$$\text{(8-10)} \quad E(\tilde{r}_p) = \bar{r}_p = E(w_1 \tilde{r}_1 + w_2 \tilde{r}_2) = w_1 \bar{r}_1 + w_2 \bar{r}_2 = \sum_{i=1}^{2} w_i \bar{r}_i$$

And hence, unexpected portfolio return is

$$\text{(8-11)} \quad \tilde{r}_p - \bar{r}_p = (w_1 \tilde{r}_1 + w_2 \tilde{r}_2) - (w_1 \bar{r}_1 + w_2 \bar{r}_2) = w_1(\tilde{r}_1 - \bar{r}_1) + w_2(\tilde{r}_2 - \bar{r}_2)$$

And thus, the variance of the portfolio[6] is:

$$\text{(8-12)} \quad \text{var}(\tilde{r}_p) = \sigma_p^2 = E\left[(\tilde{r}_p - \bar{r}_p)^2\right] = w_1^2 \sigma_1^2 + 2w_1 w_2 \sigma_{12} + w_2^2 \sigma_2^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j \sigma_{ij}$$

For example, suppose the DM invests \$600 in asset 1 ($w_1 = 0.6$) and \$400 in asset 2 ($w_2 = 0.4$) for a month. The mean (expected value) of portfolio return is $\bar{r}_p = 0.6(1.49\%) + 0.4(1.00\%) = 1.29\%$

---

[5] Using matrices (vectors) equation (8-10) can be written as $\mathbf{w}^T \bar{\mathbf{r}} = [w_1 \ w_2] \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \end{bmatrix}$.

[6] Using matrices (vectors) equation (8-12) can be written as $\mathbf{w}^T \mathbf{\Sigma} \mathbf{w} = [w_1 \ w_2] \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

and the variance of the portfolio is $\sigma_p^2 = 0.6^2(77.70\%) + 2(0.6)(0.4)(20.95\%) + 0.4^2(35.87\%) = 43.776\%$ or $\sigma_p = 6.62\%$ (standard deviation).

Using equations (8-10) and (8-12) we calculate mean return ($\bar{r}_p$) and standard deviation (σ_p) of the portfolio as in Table 8-1 with various values of w_1 and w_2. For example, when $w_1 = 0$, then $w_2 = 1$ due to the fact that $w_1 + w_2 = 1$. In this case, mean return is $\bar{r}_p = 0(1.49\%) + 1(1.00\%) = 1.00\%$ and the variance is $\sigma_p^2 = 0^2(77.70\%) + 2(0)(1)(20.95\%) + 1^2(35.87\%) = 35.87\%$ or $\sigma_p = 5.99\%$ (which is the same as the asset 2)(Table 8-1). Similarly,

- $w_1 = 0.1$ and $w_2 = 0.9$: $\bar{r}_p = 0.1(1.49\%) + 0.9(1.00\%) = 1.05\%$, $\sigma_p^2 = 0.1^2(77.70\%) + 2(0.1)(0.9)(20.95\%) + 0.9^2(35.87\%) = 33.60\%$ or $\sigma_p = 5.80\%$.

- $w_1 = 0.5$ and $w_2 = 0.5$: $\bar{r}_p = 0.5(1.49\%) + 0.5(1.00\%) = 1.25\%$, $\sigma_p^2 = 0.5^2(77.70\%) + 2(0.5)(0.5)(20.95\%) + 0.5^2(35.87\%) = 38.87\%$ or $\sigma_p = 6.23\%$.

Figure 8-1 presents the plot of mean and standard deviation of the portfolio in Table 8-1. Note that given an expected return, the portfolio that minimizes risk (measured by standard deviation) is a mean-standard deviation frontier portfolio. The locus of all frontier portfolio in the mean-standard deviation plane is called portfolio frontier (Figure 8-1). In Figure 8-1, a dot denotes the minimum variance portfolio and the upper-right part of the portfolio frontier gives efficient frontier portfolios.

Table 8-1: Portfolio of Asset 1 and Asset 2

Weight in asset 1 (%)	0	10	20	30	40	50	60	70	80	90	100
Mean return (%)	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.34	1.39	1.44	1.49
Std. Dev (%)	5.99	5.80	5.72	5.78	5.95	6.23	6.62	7.08	7.61	8.19	8.81
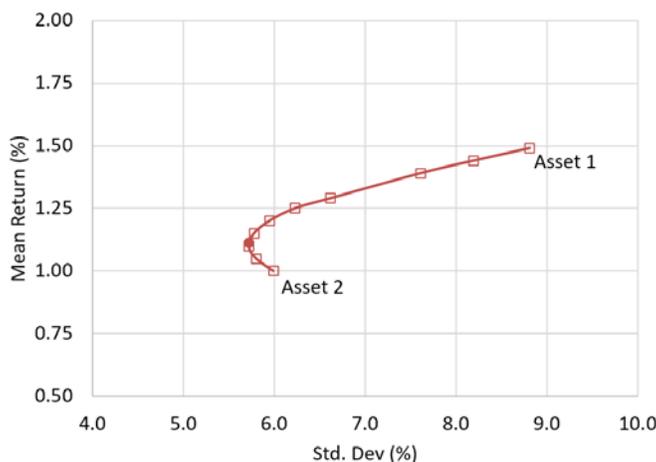
Figure 8-1. Portfolio Frontiers

8.3.1.1. *Optimal Portfolio Selection*

How to choose a portfolio? What constitutes a desirable portfolio? The primary factors we shall consider are the financial objectives of the investor and his or her *tolerance for risk* in achieving these objectives. We may want to minimize the risk measured in variance of the portfolio, σ_p^2 for a given expected return (Markowitz problem). Formally, we need to solve the following problem for minimizing risk for a given expected return:

$$
\begin{aligned}
\min \quad \sigma_p^2 &= w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2 \\
\text{s.t.} \quad & w_1 + w_2 = 1 \qquad\qquad (1) \\
& w_1\bar{r}_1 + w_2\bar{r}_2 = \bar{r}_p \qquad (2) \\
& w_1, \ w_2 \geq 0
\end{aligned}
$$

(8-13)

Or,

$$
\begin{aligned}
\min \quad \sigma_p^2 &= 77.70w_1^2 + 2\cdot 20.95w_1w_2 + 35.87w_2^2 \\
\text{s.t.} \quad & w_1 + w_2 = 1 \qquad\qquad\qquad (1) \\
& 1.49w_1 + 1.00w_2 = \bar{r}_p \qquad (2) \\
& w_1, \ w_2 \geq 0
\end{aligned}
$$

(8-14)

Solving equation (8-14) generates the portfolio frontier in Figure 8-1 as well. Note that it is a quadratic programming model and the Excel Solver with GRG Nonlinear option should be used.

8.3.2. *Portfolio of Multiple Assets*

We now consider the general case of n assets. The returns on the n assets, $\{\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_n\}$, are characterized by their mean, variance and covariances. Consider the portfolio with w_i being the proportion of the value invested in asset i. Then

(8-15) $\quad \displaystyle\sum_{i=1}^{n} w_i = 1$

The return on the portfolio, $\tilde{r}_p$, is a weighted average of the individual returns, that is,

(8-16) $\quad \displaystyle\tilde{r}_p = \sum_{i=1}^{n} w_i\tilde{r}_i$

The expected return on the portfolio is

(8-17) $\quad \displaystyle E(\tilde{r}_p) = \bar{r}_p = \sum_{i=1}^{n} w_i\bar{r}_i$

The variance of the portfolio is:

$$(8\text{-}18) \quad \mathrm{var}(\tilde{r}_p) = \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}, \quad \text{where } \sigma_{ii} = \sigma_i^2$$

Minimizing risk for a given expected return is now

$$
\begin{aligned}
\min \quad \sigma_p^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{n} w_i = 1 \qquad (1) \\
& \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}_p \qquad (2) \\
& w_i \geq 0
\end{aligned}
$$

(8-19)

Or, using matrices where $\mathbf{1} = [1\ \ 1\ \cdots\ 1]$.

$$
\begin{aligned}
\min \quad \sigma_p^2 &= \mathbf{w}^T \Sigma \mathbf{w} \\
\text{s.t.} \quad & \mathbf{w}^T \mathbf{1} = 1 \qquad (1) \\
& \mathbf{w}^T \bar{\mathbf{r}} = \bar{r}_p \qquad (2) \\
& w_i \geq 0
\end{aligned}
$$

(8-20)

Equation (8-19) or (8-20) is a portfolio which has the least variance σ_p^2 for a given expected return $\bar{r}_p$. Finding such a portfolio is referred to as the Markowitz problem and can be stated mathematically as the nonlinear constrained optimization problem. We use Excel Solver with GRG Nonlinear to solve the question numerically.

8.3.2.1. _Example with Three Assets_

Let's say we have three assets, $\tilde{r}_1$, $\tilde{r}_2$ and $\tilde{r}_3$ and suppose that the covariance matrix is

$$(8\text{-}21) \quad \Sigma = \begin{bmatrix} 77.70 & 20.95 & 11.89 \\ 20.95 & 35.87 & 2.29 \\ 11.89 & 2.29 & 97.90 \end{bmatrix}$$

And mean returns are $\{\bar{r}_1, \bar{r}_2, \bar{r}_3\} = \{1.49, 1.00, 3.00\}$. The variance of the portfolio is

$$(8\text{-}22) \quad \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = [w_1 \quad w_2 \quad w_3] \begin{bmatrix} 77.70 & 20.95 & 11.89 \\ 20.95 & 35.87 & 2.29 \\ 11.89 & 2.29 & 97.90 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Thus, equation (8-19) becomes

$$\min \quad \sigma_p^2 = 77.70w_1^2 + 35.87w_2^2 + 97.90w_3^2 + 2 \cdot 20.95w_1w_2 + 2 \cdot 11.89w_1w_3 + 2 \cdot 2.29w_2w_3$$

(8-23)
$$\text{s.t.} \quad w_1 + w_2 + w_3 = 1$$
$$1.49w_1 + 1.00w_2 + 4.20w_3 = \bar{r}_p$$
$$w_1, \quad w_2, \quad w_3 \geq 0$$

Figure 8-2 presents the portfolio frontier with three assets. Note that when more assets are included, the portfolio frontier improves, i.e., moves toward upper-left (brown line in Figure 8-2): higher mean returns and lower risk.
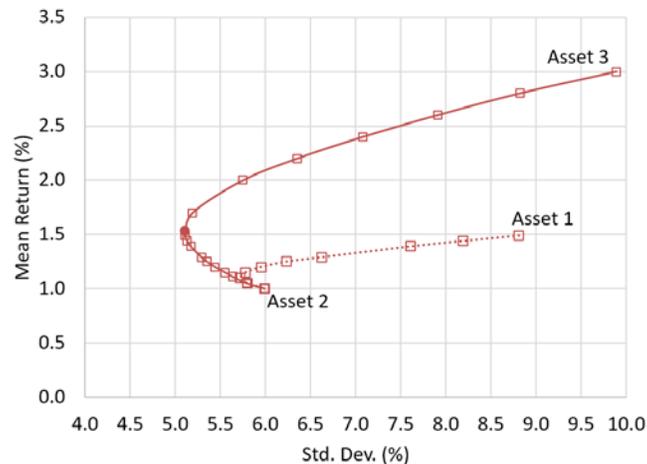
Figure 8-2. Portfolio Frontiers

8.3.2.2. *Excel Formulation*

Figure 8-3 presents the portfolio Excel formulation for three assets. Data part includes all the information we need to run the model, mean returns and variance-covariance matrix. Note that the set objective in Solver Parameters window is $C$15 where the cell includes MMULT(array1, array2) and TRANS(array) functions. MMULT function returns the matrix product of two arrays; the number of columns in array1 (C12:E12, 3 columns, 1 rows, 3×1 vector) must be the same as the number of rows in array2 (G5:I7, 3 rows, 3 columns, 3×3 matrix). TRANS function returns a transposed range of the cells. Thus the formula in the cell C15, = MMULT(MMULT(C12:E12, G5:I7), TRANS(C12:E12)) is equivalent to $\sigma_p^2 = \mathbf{w}^T\mathbf{\Sigma}\mathbf{w}$ in equation (8-15) and it is the target cell. Constraints are constructed very similar to previous Excel models.

To generate the portfolio frontier, change the mean return in the cell I20, currently it is set to be 2% in Figure 8-3, and rerun the model. Note that the solving method is GRG Nonlinear because the objective function (variance of the portfolio) is nonlinear.

Figure 8-3. Portfolio Excel Formulation

8.3.3. Mean-Variance Analysis

The portfolio approach in section 8.2 requires a given mean return of the portfolio, that is, it is not determined from the model. Mean-variance (or E-V) portfolio involves development of an optimal investment strategy including the mean of the return. The E-V model is given in equation (8-13) in case of two assets:

$$\max \quad z = w_1\bar{r}_1 + w_2\bar{r}_2 - \phi(w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2)$$

(8-24) $\quad$ s.t. $\quad w_1 + w_2 = 1$

$$w_1, \ w_2 \geq 0$$

Here the objective function maximizes mean return less a risk aversion coefficient (RAC), ϕ, times the variance of the portfolio. The E-V model assumes that a decision maker trades the mean return with reduced variance. In this context the equation (8-24) provides the E-V efficient frontier which is the locus of points exhibiting minimum variance for a given expected income, and/or maximum expected income for a given variance of portfolio. Such points are efficient for a decision maker with positive preference for income, negative preference for variance and indifference to other factors.

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For the E-V model one needs to specify RAC, ϕ. Basically, RAC is a quantitative measure of how averse to risk a person is (attitude to risk). The simplest measure of RAC is Pratt's measure[7] which is defined as

$$(8\text{-}25) \quad \phi_{ARAC} = -\frac{U''(M)}{U'(M)}$$

where M is individual (current) wealth and U is utility function. The value ϕ in equation (8-20) is called as absolute risk aversion coefficient (ARAC). Note that ARAC is decreasing with increases in M since people can better afford to take risks as they get richer. Also ARAC may depends on the monetary units of M, thus ARAC in different currency units are not comparable. Relative risk aversion coefficient (RRAC) to overcome these problems, which is defined by

$$(8\text{-}26) \quad \phi_{RRAC} = -M\frac{U''(M)}{U'(M)} = M \cdot \phi_{ARAC}$$

For the empirical analysis we may use rough and ready classification of degree of risk aversion:

- $\phi_{RRAC} = 0$ Risk neutral
- $\phi_{RRAC} = 1$ Somewhat risk averse (normal)
- $\phi_{RRAC} = 2$ Rather risk averse
- $\phi_{RRAC} = 3$ Very risk averse
- $\phi_{RRAC} = 4$ Extremely risk averse

If ϕ_{ARAC} is required, it may be derived using the formula, $\phi_{ARAC} = \phi_{RRAC}/M$. Note that, in the E-V model, we use ϕ_{ARAC}.

The E-V formulation of two assets example above is given by,

$$(8\text{-}27) \quad \begin{aligned} \max \quad z = {} & 1.49w_1 + 1.00w_2 - \phi_{ARAC}(77.70w_1^2 + 2 \cdot 20.95w_1w_2 + 35.87w_2^2) \\ \text{s.t.} \quad & w_1 + w_2 = 1 \\ & w_1, \ w_2 \geq 0 \end{aligned}$$

In general, with n assets, the E-V model is

$$(8\text{-}28) \quad \begin{aligned} \max \quad z = {} & \sum_{i=1}^{n} w_i \bar{r}_i - \phi_{ARAC} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i = 1 \\ & w_i \geq 0 \end{aligned}$$

---

[7] Pratt, J.W. (1964) "Risk Aversion in the Small and in the Large." *Econometrica* 32:122-136.

Or, equivalently

$$
\begin{array}{rl}
\max & z = \mathbf{w}^\mathrm{T}\bar{\mathbf{r}} - \phi_{\mathrm{ARAC}}\mathbf{w}^\mathrm{T}\Sigma\mathbf{w} \\
\text{(8-29)} \quad \text{s.t.} & \mathbf{w}^\mathrm{T}\mathbf{1} = 1 \\
& w_i \geq 0
\end{array}
$$

E-V Example: Assume an investor wishes to develop a stock portfolio given the stock annual returns information shown in Table 8.2; 500 dollars to invest and prices of stock one $22, stock two $30, stock three $28 and stock four $26, respectively. The first stage in model application is to compute average returns and the variance-covariance matrix of total net returns. The mean returns are

$$
\text{(8-30)} \quad \bar{\mathbf{r}} = \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \\ \bar{r}_4 \end{bmatrix} = \begin{bmatrix} 4.7 \\ 7.6 \\ 8.3 \\ 5.8 \end{bmatrix}
$$

And, the covariance matrix is

$$
\text{(8-31)} \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_4^2 \end{bmatrix} = \begin{bmatrix} 3.21 & -3.52 & 6.99 & 0.04 \\ -3.52 & 5.84 & -13.68 & 0.12 \\ 6.99 & -13.68 & 61.81 & -1.64 \\ 0.04 & 0.13 & -1.64 & 0.36 \end{bmatrix}
$$

Table 8-2: Data for E-V Example – Monthly Returns by Stock (**dollars/stock**)

Month	Stock 1	Stock 2	Stock 3	Stock 4
1	7	6	8	5
2	8	4	16	6
3	4	8	14	6
4	5	9	-2	7
5	6	7	13	6
6	3	10	11	5
7	2	12	-2	6
8	5	4	18	6
9	4	7	15	5
10	3	9	-5	6
Price of stock	22	30	28	26

Let the decision variables are the amount of stock to buy, v_i, or,

$$(8\text{-}32) \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

And, in turn, the objective function is $z = \mathbf{v}^T \bar{\mathbf{r}} - \phi_{ARAC} \mathbf{v}^T \Sigma \mathbf{v}$ or,

$$(8\text{-}33) \quad z = [v_1\ v_2\ v_3\ v_4] \begin{bmatrix} 4.7 \\ 7.6 \\ 8.3 \\ 5.8 \end{bmatrix} - \phi_{ARAC}[v_1\ v_2\ v_3\ v_4] \begin{bmatrix} 3.21 & -3.52 & 6.78 & 0.04 \\ -3.52 & 5.84 & -13.86 & 0.12 \\ 6.78 & -13.86 & 61.84 & -1.88 \\ 0.04 & 0.12 & -1.88 & 0.36 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

This objective function is maximized subject to a constraint on investable funds:

$$(8\text{-}34) \quad 22v_1 + 30v_2 + 28v_3 + 26v_4 \leq 500$$

and non-negativity conditions on the variables.

Excel formulation is given by Figure 8-4.

Figure 8-4. E-V Model: Portfolio Optimization

The model yields the z (difference of mean and RAC × variance) maximizing solution ($v_1 = v_2 = v_4 = 0$ and $v_3 = 17.86$) for the risk neutral decision maker ($\phi = 0$) with expected return of $144 and standard deviation of $154 (point with ARAC=0 in Figure 8-4). As the risk aversion parameter increases, then v_2 comes into the solution. The simultaneous use of v_2 and v_3 coupled with their negative covariance reduces the variance of total returns. This pattern continues as ϕ increases. For example, when $\phi = 1$, expected return has fallen to $118 with a low standard deviation of $6.77 and the solution is $v_1 = 3.78$; $v_2 = 4.95$, $v_3 = 0.99$, and $v_4 = 9.27$ (point with ARAC=1 in Figure 8-5). Figure 8-5 plots the efficiency frontier which is a line created from the risk-reward graph, comprised of optimal portfolios.
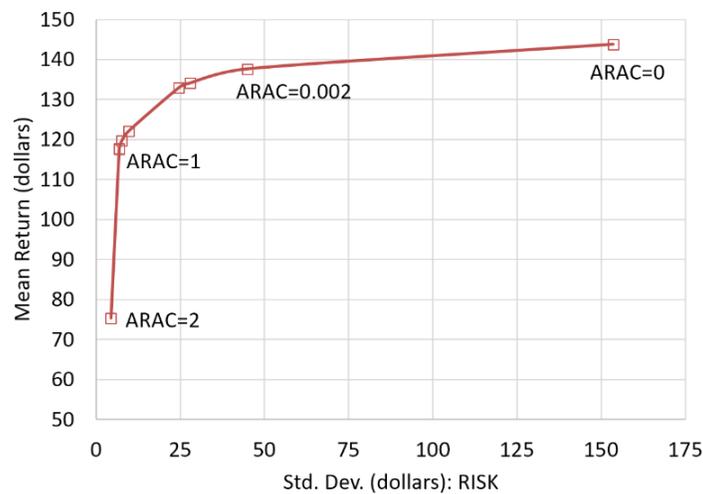
Figure 8-5. Efficiency Frontier

A portfolio above the curve is impossible and a portfolio below the curve is not efficient, because for the same risk (same standard deviation) one could achieve a greater return. The optimal portfolio should line on the curve (so as it is the efficient frontier).

Three aspects of the results are worth noting. First, the shadow price on investable capital (from the Sensitivity Report) continually decreases as the risk aversion parameter, ϕ, increases. This reflects an increasing risk discount as risk aversion increases. Second, solutions are reported only for selected values of ϕ. However, any change in ϕ leads to a different solution and an infinite number of alternative ϕ's are possible; e.g., all solutions between ϕ values of, for example, 0.5 and 0.75 are convex combinations of those two solutions. Third, when ϕ becomes sufficiently large, the model does not use all its resources. In this particular case, when ϕ exceeds 2.5, not all funds are invested.

9. RISK MODELING AND STOCHASTIC PROGRAMMING

Key points:

Risk modeling or stochastic programming is techniques which are designed to give a robust solution which yields satisfactory results across the full distribution of parameter values.

Two types of risk modeling: without recourse and with recourse

Models without recourse deal with uncertainties in
- Objective coefficients: mean-variance (E-V model)
- RHS: chance-constrained programming
- Technical coefficients: Merrill's approach

Model with recourse (stochastic programming with recourse (SPR)) allows to deal with sequential risk and adaptive decision making.

9.1. Introduction

So far we assume that each coefficient of the objective function and constraints is known for sure (not random) but sometimes this is not true. Some outcomes, such as crop yields or prices, are not known with certainty in the planting season. These imperfect knowledge is uncertainty. Because of uncertainty, we are now exposed to risk which is defined as uncertain consequences, particularly unfavorable consequences. Mathematical programming is very flexible to incorporate risk. Note that the portfolio analysis in Chapter 8 is the one way to deal with it.

Risk must be quantified to evaluate alternative decisions. The measuring of uncertainty involves estimating the probabilities of (future) outcomes. To estimate such probabilities we generally start by observing historical outcomes and separating random variability from systematic variability. This is summarized into coefficient of variation and/or probability density distribution (continuous variable) or probability mass function (discrete variable).

Risk modeling or stochastic programming is techniques which are designed to give a robust solution which yields satisfactory results across the full distribution of parameter values. The risk modeling techniques discussed below are designed to yield such a plan. The optimal plan for a stochastic model generally does not place the decision maker in the best possible position for all (or maybe even any) possible parameter combinations(commonly called states of nature or events), but rather establishes a robust position across the set of possible events.

9.2. Decision Making and Recourse

Many different stochastic programming formulations have been posed for risk problems. An important assumption involves the potential decision maker reaction to information. The most fundamental distinction is between cases where:

- All decisions must be <u>made now</u> with the uncertain outcomes resolved later, after all random draws from the distribution have been taken, and

- Some decisions are made now, then later some uncertainties are resolved <u>followed by other decisions later</u>.

These two settings are illustrated as follows. In the first case, all decisions are made then events occur and outcomes are realized. This is akin to a situation where one invests now and then discovers the returns to the investment at years end without any intermediate buying or selling decisions (recall the portfolio examples in Chapter 8). In the second case, one makes some decisions now, gets some information and makes subsequent decisions. Thus, one might invest at the beginning of the year based on a year-long consideration of returns, but could sell and buy during the year depending on changes in stock prices.

The main distinction is that under the first situation decisions are made before any uncertainty is resolved and no decisions are made after any of the uncertainty is resolved. In the second situation, decisions are made sequentially with some decisions made conditional upon outcomes that were subject to a probability distribution at the beginning of the time period.

These two frameworks lead to two very different types of risk programming models. The first type of model is most common and is generally called a stochastic programming model <u>without recourse</u>. The second type of model falls into the class of stochastic programming <u>with recourse</u> models. These approaches are discussed separately, although many of the "without recourse" techniques can be used when dealing with the "with recourse" problems.

9.3. Stochastic Programming without Recourse

Risk may arise in the objective function coefficients, technical coefficients, or right hand sides separately or collectively. Different modeling approaches have arisen with respect to each of these possibilities and we will cover each separately.

9.3.1. Objective Function Coefficient Risk

Given a linear objective function with two decision variables $z = c_1 x_1 + c_2 x_2$ where x_1, x_2 are decision variables and c_1, c_2 are uncertain parameters (profit margins, for example) with means $\bar{c}_1$ and $\bar{c}_2$ as well as variances σ_1^2, σ_2^2 and covariance σ_{12}; then z is distributed with mean

$$(9\text{-}1) \quad \bar{z} = \bar{c}_1 x_1 + \bar{c}_2 x_2 = \sum_j \bar{c}_j x_j = [\bar{c}_1 \quad \bar{c}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \bar{\mathbf{c}}^T \mathbf{x}$$

and variance

$$(9\text{-}2) \quad \sigma_z^2 = \sigma_1^2 x_1^2 + 2\sigma_{12} x_1 x_2 + \sigma_2^2 x_2^2 = \sum_j \sum_k \sigma_{jk} x_j x_k = [x_1 \quad x_2] \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T \mathbf{S} \mathbf{x}$$

Similar to the portfolio analysis in Chapter 8, we formulate Mean-Variance model:

$$\max \quad z = \sum_{j=1}^{n} \bar{c}_j x_j - \phi_{ARAC} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{ij} x_j x_k$$

(9-3) s.t.

$$\sum_{i=1}^{m} a_{ij} x_j \qquad\qquad \leq b_i \quad \text{for all i}$$

$$x_j \qquad\qquad \geq 0 \quad \text{for all j}$$

where ϕ_{ARAC} is a risk aversion coefficient (RAC). The objective function maximizes expected income less RAC times the variance of total income (risk); the model assumes that decision makers will trade the expected income for the reduced variance. See Section 8.3.3 and example for more discussion regarding E-V formulation.

EV Model Example: A farmer has three crops to plant (in the planting season) and net return (profit margins) over five years are given in Table 9-1. Other information is given as following

- The farm has 12 acres of cropland,
- No more than 6 acres can be sown in total of crop 1 and crop 3 due to crop rotation,
- There is \$400 of working capital available and crop 1 takes \$30/acre, crop 2 takes \$20/acre and crop 3 takes \$40/acre,
- There are 80 hours of labor in the planting season; crop 1 needs 5 hours/acre, crop 2 requires 5 hours/acre and crop 3 needs 8 hours/acre.

Finding means ($\bar{c}_j$):

$$\bar{c}_j = \frac{\sum c_{jt}}{N} \quad \text{use} = \text{AVERAGE(range) in Excel}$$

$$\bar{c}_1 = \frac{99 + 133 + 143 + 154 + 114}{5} = 128.6$$

$$\bar{c}_2 = \frac{118 + 130 + 133 + 127 + 95}{5} = 120.6$$

$$\bar{c}_3 = \frac{65 + 61 + 55 + 58 + 69}{5} = 61.6$$

Variances and covariance

$$\sigma_j^2 = \frac{\sum(c_{jt} - \bar{c}_j)^2}{N} \quad \text{and} \quad \sigma_{ij} = \frac{\sum(c_{it} - \bar{c}_i)(c_{jt} - \bar{c}_j)}{N} \quad \text{use} = \text{COVAR(range i, range j) in Excel}$$

$$\sigma_1^2 = \frac{(99 - 128.6)^2 + (133 - 128.6)^2 + \cdots + (114 - 128.6)^2}{5} = 392.24$$

Table 9-1: Data for E-V Example – Net Return

Year	Net return ($/acre)		
	Crop 1	Crop 2	Crop 3
1	99	118	65
2	133	130	61
3	143	133	55
4	154	127	58
5	114	95	69
Mean	128.6	120.6	61.6

$$\sigma_2^2 = \frac{(118 - 120.6)^2 + (130 - 120.6)^2 + \cdots + (95 - 120.6)^2}{5} = 189.04$$

$$\sigma_2^2 = \frac{(65 - 61.6)^2 + (61 - 61.6)^2 + \cdots + (69 - 61.6)^2}{5} = 189.04$$

$$\sigma_{12} = \frac{(99 - 128.6)(118 - 120.6) + \cdots + (114 - 128.6)(95 - 120.6)}{5} = 166.64 = \sigma_{21}$$

$$\sigma_{13} = \frac{(99 - 128.6)(65 - 61.6) + \cdots + (114 - 128.6)(69 - 61.6)}{5} = -79.56 = \sigma_{31}$$

$$\sigma_{23} = \frac{(118 - 120.6)(65 - 61.6) + \cdots + (95 - 120.6)(69 - 61.6)}{5} = -61.76 = \sigma_{32}$$

Thus, the EV formulation is

$$\max \quad z = [128.6 \ 120.6 \ 61.6] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \phi[x_1 \ x_2 \ x_3] \begin{bmatrix} 392.24 & 166.64 & -79.56 \\ 166.64 & 189.04 & -61.76 \\ -79.56 & -61.76 & 24.64 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(9-4) s.t.

$$x_1 + x_2 + x_3 \leq 12 \qquad\qquad\qquad\qquad \text{[Land]}$$
$$30x_1 + 20x_2 + 40x_3 \leq 400 \qquad\qquad \text{[Capital]}$$
$$5x_1 + 5x_2 + 8x_3 \leq 80 \qquad\qquad\quad \text{[Labor]}$$
$$x_1 + x_3 \leq 6 \qquad\qquad\qquad\qquad\quad \text{[Rotation]}$$

And Excel formulation is presented in Figure 9-1. As we did in Chapter 8, this problem is solved for various ϕ values and the efficiency frontier is constructed. An efficiency frontier is a line created from the risk-reward graph, comprised of optimal portfolios. Refer to the discussion is Section 8.3.3 for the specification of ϕ.

Figure 9-1. E-V Formulation

9.3.2. Right Hand Side Risk

Risk may also occur within the right hand side (RHS) parameters; the most often used approach to RHS risk in a non-recourse setting is chance-constrained programming (CCP). In this formulation, ith constraint with stochastic RHS is given by (assuming we have two decision variables)

(9-5) $\quad a_{i1}x_1 + a_{i2}x_2 \leq \bar{b}_i - z_\alpha \sigma_i$

where $\bar{b}_i$ is the mean of (historical) resource i availability, σ_i is the standard deviation of resource i availability, and α is the predetermined (desired) value of probability such that $Pr(a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \leq b_i) \geq \alpha$, that is, resource use must be less than or equal to average resource availability less the standard deviation times a critical value which arises from the probability level. Values of z_α may be determined in two ways

- By making normality assumption about the form of the probability distribution of b_i, use values for the lower tail from a standard normal probability table, for example, use $z_{90\%} = 1.28$ (meaning that $Pr(z \geq -1.28) = 10\%$), $z_{95\%} = 1.64$ or $z_{99\%} = 2.33$, respectively.

- By relying on the conservative estimates generated by using Chebyshev's inequality such that $z_\alpha = (1 - \alpha)^{-0.5}$. For example, $z_{90\%} = (1 - 0.9)^{-0.5} \approx 3.16$. Note that Chebyshev bound is often too large.

When the RHS values vary about the mean values, i.e., it is random, the solution is feasible only about certain percent of the time. The idea of CCP is that one way to find a solution that has a greater feasibility probability is to make the RHS vector *smaller*.

Example – RHS Risk

Let's recall Joe's van example and say labor availability is uncertain (for some reasons), and labor availability for the past 6 weeks are 280, 283, 294, 274, 260, and 289 hours. Average of the observations, $\bar{b}_{labor}$, is 280 and standard deviation, σ_{labor}, is 12.02. The chance-constrained programming model is

$$
\begin{array}{llrclll}
\text{max} & z = & 2000x_{fancy} & + & 1700x_{fine} & & \\
\text{s.t.} & & x_{fancy} & + & x_{fine} & \leq 12 & \text{[capacity constraint]} \\
& & 25x_{fancy} & + & 20x_{fine} & \leq 280 - z_\alpha \cdot 12.02 & \text{[labor constraint]} \\
& & x_{fancy} & , & x_{fine} & \geq 0 & \text{[non-negativity]}
\end{array}
$$

(9-6)

Table 9-2 presents the results assuming normality, that is, the distribution of the labor availability is normal. Note that, without labor risk, $z_\alpha = 0$. Notice as the z_α value is increased, then the value of the uncertain right hand side decreases. In turn, composition of production changes (produce more fine van which requires less labor) and profit decreases (Table 9-2). The chance constrained model discounts the resources available, so one is more certain that the constraint will be met.

The major advantage of CCP is its simplicity; it leads to an equivalent programming problem of about the same size and the only additional data requirements are the standard errors of the RHS. Despite the fact that CCP is a well-known technique and has been applied its use has been limited and controversial. A fundamental problem with CCP is that it does not indicate what to do if the recommended solution is not feasible.

Table 9-2: Joe's Van with RHS (Labor) Risk

		Labor available	Fancy van	Fine van	Profit
$z_\alpha = 0$	No risk	280	8.00	4.00	$22,800
$z_\alpha = 1.28$	90% confidence level	264.62	4.92	7.08	$21,877
$z_\alpha = 1.64$	95% confidence level	260.29	4.06	7.94	$21,618
$z_\alpha = 1.96$	95% confidence level	256.45	3.29	8.71	$21,387

9.3.3. Technical Coefficient Risk

Risk can also appear within the technical coefficients. Resolution of technical coefficient uncertainty in a non-recourse setting has been investigated through an E-V like procedure (Merrill's approach).

Let's recall Joe's van example and say the labor requirements, 25 hours for fancy van and 20 hours for fine van, are uncertain for some reasons, e.g., different quality of materials which may need more time to work, etc. The labor requirements for past five weeks are given in Table 9-3 with mean and variance-covariance matrix.

Table 9-3: Joe's Van Labor Requirements

Week	Fancy van (hours/van)	Fine van (hours/van)	Variance-covariance	
			Fancy	Fine
1	24	20	1.60	0.20
2	27	19	0.20	2.00
3	24	21		
4	26	22		
5	24	18		
Mean	**25**	**20**		

In Merrill's approach, we include the mean and variance of the technical coefficients into the constraint (like E-V model) such that $\sum \bar{a}_{ij}x_j + z_\alpha(\sum\sum \sigma_{ijk}x_jx_k)$, where $\bar{a}_{ij}$ is the mean value of the (past) a_{ij}'s and σ_{ijk} is the covariance of the a_{ij} and a_{ik}, and z_α is the desired value of the distribution (upper tail for the less than equal to constraint). Note that $\sigma_{ijj} = \sigma_{ij}^2$. The Joe's van formulation becomes

$$
\begin{array}{llllll}
\max & z = & 2000x_{fancy} & + & 1700x_{fine} & \\
\text{s.t.} & & x_{fancy} & + & x_{fine} & \leq \quad 12 \\
& & 25x_{fancy} & + & 20x_{fine} + z_\alpha(1.60x_{fancy}^2 + 2.00x_{fine}^2 + 0.4x_{fancy}x_{fine}) & \leq \quad 280 \\
& & x_{fancy} & , & x_{fine} & \geq \quad 0
\end{array}
$$
(9-7)

Table 9-3 presents the results assuming normality of labor requirements (technical coefficients). The idea of Merrill's approach that one way to find a solution that has a greater labor requirements by adding variance term.

Table 9-4: Joe's Van with RHS (Labor) Risk

		Fancy van	Fine van	Profit
$z_\alpha = 0$	No tech coefficient risk	8.00	4.00	$22,800
$z_\alpha = 1.28$	90% confidence level	4.94	3.41	$15,674
$z_\alpha = 1.64$	95% confidence level	4.67	3.17	$14,727
$z_\alpha = 1.96$	95% confidence level	4.46	3.01	$14,033

9.4. Stochastic Programming with Recourse: Sequential Risk

Sequential risk arises as part of the risk as time goes on and adaptive decisions are made. Consider the way that weather and field working time risks are resolved in crop farming. Early on, planting and harvesting weather are uncertain. After the planting season, the planting decisions have been made and the planting weather has become known, but harvesting weather is still uncertain. Under such circumstances a decision maker would <u>adjust</u> to conform to the planting pattern but would still need to make harvesting decisions in the face of harvest time uncertainty. Thus sequential risk models must depict adaptive decisions along with fixity of earlier decisions (a decision maker cannot always undo earlier decisions such as planted acreage). Nonsequential risk, on the other hand, implies that a decision maker chooses a decision now and finds out about all sources of risk later. All the models above are nonsequential risk models. Stochastic programming with recourse (SPR) models are used to depict sequential risk. We will discuss two-stage LP formulation.

9.4.1. Two Stage SPR

Let us consider a simple farm planning problem. Suppose we can raise corn and wheat on a 100 acre farm. Suppose per acre planting cost for corn is $100 while wheat costs $60. However, suppose crop yields, harvest time requirements per unit of yield, harvest time availability and crop prices are uncertain. To make the problem simple, suppose that there will be two state of nature (SON) 1 and 2 as indicated in Table 9-5.

Table 9-5: Data on Uncertain Parameters in SPR Example

Parameter	State of nature 1 (SON1) (e.g., dry harvesting season)	State of nature 2 (SON2) (e.g., wet harvesting season)
Probability	0.6	0.4
Planting cost per acre ($)	100	60
Corn yield (bushel)	100	105
Wheat yield (bushel)	40	38
Corn harvest rate hours per bu	0.010	0.015
Wheat harvest rate hours per bu	0.030	0.034
Corn price	3.25	2.00
Wheat price	5.00	6.00
Harvest time hours	122	143

Let $x_{p,c}$ be the acre of planting corn, $x_{p,w}$ the acre of planting wheat, $x_{h,c}$ the corn production and $x_{h,w}$ the wheat production, and y the income. The deterministic problem assuming Pr(SON1) = 1 is given in equation (9-8) and the optimal solution is to produce corn only where y = 22500, $x_{p,c}$ = 100 acres, $x_{h,crn}$ = 10000 bu., and $x_{p,w} = x_{h,w} = 0$. 100 hours of harvest time is used. What if Pr(SON2) = 1? Equation (9-9) presents the model and the solution is to produce wheat only where y = 16800, $x_{p,c}$ = $x_{h,c} = 0$, $x_{p,w} = 100$, $x_{h,w} = 3800$ bu.; 129.2 hours of harvest time is used. The solution is not robust.

$$
\begin{array}{lrrrrrrrl}
\max & y \\
\text{s.t.} & -y & -100x_{p,c} & -60x_{p,w} & +3.25x_{h,c} & +5.00x_{h,w} & = & 0 & \text{[Income]} \\
& & -100x_{p,c} & & +x_{h,c} & & = & 0 & \text{[corn yield balance]} \\
(9\text{-}8) & & & -40x_{p,w} & & +x_{h,w} & = & 0 & \text{[wheat yield balance]} \\
& & & & 0.01x_{h,c} & +0.03x_{h,w} & \leq & 122 & \text{[Harvest time]} \\
& & x_{p,c} & +x_{p,w} & & & \leq & 100 & \text{[Land]} \\
& & x_{p,c} \;, & x_{p,w} \;, & x_{h,c} \;, & x_{h,w} & \geq & 0 & \text{[Non-negativity]}
\end{array}
$$

$$
\begin{array}{lrrrrrrrl}
\max & y \\
\text{s.t.} & -y & -100x_{p,c} & -60x_{p,w} & +2.00x_{h,c} & +6.00x_{h,w} & = & 0 & \text{[Income]} \\
& & -105x_{p,c} & & +x_{h,c} & & = & 0 & \text{[corn yield balance]} \\
(9\text{-}9) & & & -38x_{p,w} & & +x_{h,w} & = & 0 & \text{[wheat yield balance]} \\
& & & & 0.015x_{h,c} & +0.034x_{h,w} & \leq & 122 & \text{[Harvest time]} \\
& & x_{p,c} & +x_{p,w} & & & \leq & 100 & \text{[Land]} \\
& & x_{p,c} \;, & x_{p,w} \;, & x_{h,c} \;, & x_{h,w} & \geq & 0 & \text{[Non-negativity]}
\end{array}
$$

The SPR formulation of this example is given in equation (9-10) and Excel formulation in Figure 9-2, which maximizes expected income across the SON ($\bar{y} = 0.6y_1 + 0.4y_2$, assuming the DM is risk neural, ignoring risk) but the harvesting variable levels depend on SON. This equation contains one set of first stage variables (i.e., one set of corn growing and wheat growing activities) coupled with two sets of second stage variables after the uncertainty is resolved (i.e., there are income, harvest corn, and harvest wheat variables for both states of nature). Further, there is a single unifying objective function and land constraint, but two sets of constraints for the states of nature (i.e., two sets of corn and wheat yield balances, harvesting hour constraints and income constraints). Notice underneath the first stage corn and wheat production variables, that there are coefficients in both the state of nature dependent constraints reflecting the different uncertain yields from the first stage (i.e., corn yields 100 bushels under the first state of nature and 105 under the second; while wheat yields 40 under the first and 38 under the second). However, in the second stage resource usage for harvesting is independent. Thus, the 122 hours available under the first state of nature cannot be utilized by any of the activities under the second state of nature. Also, the crop prices under the harvest activities vary by state of nature as do the harvest time resource usages.

The example model then reflects, for example, if one acre of corn is grown that 100 bushels will be available for harvesting under state of nature one, while 105 will be available under state of nature two. In the optimal solution there are two harvesting solutions, but one production solution. Thus, we model irreversibility (i.e., the corn and wheat growing variable levels maximize expected income across the states of nature, but the harvesting variable levels depend on state of nature).

$$\max \quad \bar{y}$$

$$
\begin{array}{lll}
\text{s.t.} & -\bar{y} + 0.6y_1 + 0.4y_2 & = 0 \quad \text{[Expected income]} \\
& -y_1 - 100x_{p,c} - 60x_{p,w} + 3.25x_{1,h,c} + 5.00x_{1,h,w} & = 0 \quad \text{[Income SON1]} \\
& -100x_{p,c} + x_{1,h,c} & = 0 \quad \text{[Corn yield SON1]} \\
& -40x_{p,w} + x_{1,h,w} & = 0 \quad \text{[Wheat yield SON1]} \\
& 0.01x_{1,h,c} + 0.03x_{1,h,w} & \leq 122 \quad \text{[Harvest time SON1]} \\
& -y_2 - 100x_{p,c} - 60x_{p,w} + 2.0x_{2,h,c} + 6.0x_{2,h,w} & = 0 \quad \text{[Income SON2]} \\
& -105x_{p,c} + x_{2,h,c} & = 0 \quad \text{[Corn yield SON2]} \\
& -38x_{p,w} + x_{2,h,w} & = 0 \quad \text{[Wheat yield SON2]} \\
& 0.015x_{1,h,c} + 0.034x_{2,h,w} & \leq 122 \quad \text{[Harvest time SON2]} \\
& x_{p,c} + x_{p,w} & \leq 100 \quad \text{[Land]} \\
& x_{p,c}, \quad x_{p,w}, \quad x_{1,h,c}, \quad x_{1,h,w}, \quad x_{2,h,c}, \quad x_{2,h,w} & \geq 0 \quad \text{[Non-negativity]}
\end{array}
$$

(9-10)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1														
2								SON1		SON2				
3			INCOME			Plant	Plant	Harvest	Harvest	Harvest	Harvest			
4			Bar	INCOME 1	INCOME 2	Corn	Wheat	Corn S1	Wheat S1	Corn S2	Wheat S2			
5			16475.6	18144.88	13971.73	48.76	51.24	4876.33	2049.47	5120.14	1947.00	MAX		
6			1									16475.62		
7												USED		RHS
8	Avg Income		-1	0.6	0.4							0.00	=	0
9	State 1	Income		-1		-100	-60	3.25	5.00			0.00	=	0
10		Corn yield				-100		1.00				0.00	=	0
11		Wheat yield					-40		1.00			0.00	=	0
12		Harvest hours						0.01	0.03			110.25	<=	122
13	State 2	Income			-1	-100	-60			2.00	6.00	0.00	=	0
14		Corn yield				-105				1.00		0.00	=	0
15		Wheat yield					-38				1.00	0.00	=	0
16		Harvest hours								0.015	0.034	143.00	<=	143
17		Land				1	1					100.00	<=	100
18														

Figure 9-2. Two Stage SPR in Excel Formulation

The SPR solution tell us that the acreage is basically split 50-50 between corn and wheat, but harvesting differs with 4876 bushels of corn harvested under the first state, where as 5120 bushels of corn are harvested under the second. This shows <u>adaptive decision making</u> with the harvest decision conditional on state of nature. The model also shows different income levels by state of nature with $18,145 made under state of nature one and $13,972 under state of nature two.

The income accounting feature also merits discussion. Note that the full cost of growing corn is accounted for under both the first and second states of nature. However, since income under the first state of nature is multiplied by 0.6 and income under the second state of nature is multiplied by 0.4, then no double counting is present.

9.4.2. Incorporating Risk Aversion

The two stage model as presented above is risk neutral. This two stage formulation can be altered by adding (income) variance term with risk aversion coefficient (RAC), ϕ. To find the variance of income, we use the deviation variables, d^+ and d^-. Deviation of income under SON1 is obtained by using equation (9-11):

$$(9\text{-}11) \quad -\bar{y} + y_1 - d_1^+ + d_1^- = 0$$

For example, from Figure 9-2, we know that $\bar{y} = 16476$, $y_1 = 18145$, and $-\bar{y} + y_1 = 1669$, which implies $d_1^+ = 1669$ to satisfy equation (9-11). Similarly, deviation of income under SON2 is obtained by using equation (9-12)

$$(9\text{-}12) \quad -\bar{y} + y_2 - d_2^+ + d_2^- = 0$$

From Figure 9-2 again, we know that $\bar{y} = 16476$, $y_2 = 13972$, and $-\bar{y} + y_2 = -2504$, which implies $d_2^- = 2504$ to satisfy equation (9-12). Note that d^+ indicates income above the average whereas d^- indicates shortfalls.

Risk averse DM doesn't like the large deviations. To incorporate risk aversion, the objective function is now

$$(9\text{-}13) \quad \max \quad \bar{y} - \phi[0.6(d_1^+ + d_1^-)^2 + 0.4(d_2^+ + d_2^-)^2]$$

where $0.6(d_1^+ + d_1^-)^2 + 0.4(d_2^+ + d_2^-)^2$ is the variance of the income.

Two stage SPR with risk aversion is presented in equation (9-14) and Excel formulation in Figure (9-3).

$$\max \quad \bar{y} \qquad - \phi[0.6(d_1^+ + d_1^-)^2 + 0.4(d_2^+ + d_2^-)^2]$$

$$\text{s.t.} \quad -\bar{y} + 0.6y_1 + 0.4y_2 = 0$$
$$-\bar{y} + y_1 \qquad - d_1^+ + d_1^- = 0$$
$$-\bar{y} \qquad + y_2 \qquad - d_2^+ + d_2^- = 0$$
$$-y_1 \qquad -100x_{p,c} - 60x_{p,w} + 3.25x_{1,h,c} + 5.00x_{1,h,w} = 0$$
$$-100x_{p,c} \qquad + x_{1,h,c} = 0$$
$$-40x_{p,w} \qquad + x_{1,h,w} = 0$$
$$0.01x_{1,h,c} + 0.03x_{1,h,w} \le 122$$
$$-y_2 \qquad -100x_{p,c} - 60x_{p,w} \qquad + 2.0x_{2,h,c} + 6.0x_{2,h,w} = 0$$
$$-105x_{p,c} \qquad + x_{2,h,c} = 0$$
$$-38x_{p,w} \qquad + x_{2,h,w} = 0$$
$$0.015x_{2,h,c} + 0.034x_{2,h,w} \le 122$$
$$x_{p,c} + x_{p,w} \le 100$$
$$d_1^+, \ d_1^-, \quad d_2^+, \ d_2^-, \quad x_{p,c}, \ x_{p,w}, \quad x_{1,h,c}, \quad x_{1,h,w}, \quad x_{2,h,c}, \quad x_{2,h,w} \ge 0$$

(9-14)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1																		
2						Deviations						SON1		SON2		RAC coef		
3			INCOME							Plant	Plant	Harvest	Harvest	Harvest	Harvest	0.5		
4			Bar	INCOME 1	INCOME 2	d1+	d1-	d2+	d2-	Corn	Wheat	Corn S1	Wheat S1	Corn S2	Wheat S2			
5			15664	15665	15664	0.32	0.00	0.00	0.49	19.6	80.4	1959	3217	2056.54	3055.73	MAX		
6			1.0													15664	=C5−P3*(D8*(F5+G5)^2+E8*(H5+I5)^2)	
7																USED		RHS
8	Avg Income		-1.0	0.6	0.4											0.00 =		0
9	Income 1 deviation		-1.0	1.0		-1.0	1.0									0.00 =		0
10	Income 2 deviation		-1.0		1.0			-1.0	1.0							0.00 =		0
11	State 1	Income		-1.0						-100	-60	3.25	5.00			0.00 =		0
12		Corn yield								-100		1.00				0.00 =		0
13		Wheat yield									-40		1.00			0.00 =		0
14		Harvest hours										0.01	0.03			116.08 <=		122
15	State 2	Income			-1.0					-100	-60			2.00	6.00	0.00 =		0
16		Corn yield								-105				1.00		0.00 =		0
17		Wheat yield									-38				1.00	0.00 =		0
18		Harvest hours												0.015	0.034	134.74 <=		143
19		Land								1	1					100.00 <=		100
20																		

Figure 9-3. Two Stage SPR with Risk in Excel Formulation

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The SPR with risk solution tells us that the acreage is now split 20-80 between corn and wheat not 50-50. The expected income is 15664 which is lower than 16476 in Figure 9-2 but deviations are much smaller which indicates that the DM expects about $15665 either SON1 or SON2 (removing the risk).

The SPR model is perhaps the most satisfying of the risk models. Conceptually it incorporates all sources of uncertainty: right hand side, objective function and technical coefficients while allowing adaptive decisions.

9.5.   Comments on Risk Modeling

As demonstrated above, there are a number of ways of handling uncertainty when modeling. Several aspects of these types of models need to be pointed out. First, the modeler must assume knowledge of the distribution of risk faced by a decision maker and the risk aversion coefficient.

The second set of comments regards data. Important parameters within the context of risk models are the expectation of the coefficient value and its probability distribution around that expectation. The most common practice for specification of these parameters is to use the <u>historical mean and variance</u>. This, however, is neither necessary nor always desirable. Fundamentally, the measures that are needed are the value expected for each uncertain parameter and the perceived probability distribution of deviations from that expectation (with joint distributions among the various uncertain parameters). The parameter expectation is not always a historical mean. This is most unrealistic in cases where there has been a strong historical trend. There is a large body of literature dealing with expectations and/or time series analysis, and some use of these results and procedures appears desirable.

Data are most often generated historically; however, observations could be generated by several other means. For example, observations could be developed from a simulation model from a forecasting equation, or from subjective interrogation of the decision maker. There are cases where these other methods are more appropriate than history due to such factors as limited historical data (say, on the price of a new product) or major structural changes in markets. Naturally, the form in which the data are collected depends on the particular application involved.

A final comment on data regards their probabilistic nature. Basically when using historically based means and variance one is assuming that all observations are equally probable. When this assumption is invalid, the model is modified so that the value expected is the probabilistically weighted mean (if desired) and the variance formula includes the consideration of probability. Deviation models must also be adjusted so that the deviations are weighted by their probability as well.

A third comment relates to the question "should uncertainty be modeled and if so, how?" Such a concern is paramount to this section. It is obvious from the above that in modeling uncertainty, data are needed describing the uncertainty, and that modeling uncertainty makes a model larger and more complex, and therefore harder to interpret, explain, and deal with. It is not the purpose of these

comments to resolve this question, but rather to enter some considerations to the resolution of this question. First and fundamentally, if a model solution diverges from reality because the decision maker in reality has somehow considered risk, then it is important to consider risk. This leads to the subjective judgment on behalf of the modeling team as to whether risk makes a difference. Given that risk is felt to make a difference, then, how should risk be modeled? In the approaches above, the formulation model depends upon whether there is conditional decision making and on what is uncertain. These formulations are not mutually exclusive; rather, it may be desirable to use combinations of these formulations.